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Classical and Quantum Dynamics of Constrained Hamiltonian Systems

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Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data
Rothe, Heinz J.
Classical and quantum dynamics of constrained Hamiltonian systems / by
Heinz J Rothe & Klaus D Rothe.
p. cm. -- (World scientific lecture notes in physics ; v. 81)
Includes bibliographical references and index.
ISBN-13: 978-981-4299-64-0 (hardcover : alk. paper)
ISBN-10: 981-4299-64-2 (hardcover : alk. paper)
1. Quantum theory. 2. Constrained Hamiltonian systems. 3. Quantum field theory.
4. Gauge field theories. I. Rothe, Klaus D. (Klaus Dieter) II. Title.
QC174.125.R68 2010
530.12--dc22

2009052083

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Printed in Singapore.

To Renate and our children
Christine, Stefan and Laura

Heinz Rothe

To Neusa Maria,
and our son Thomas

Klaus Rothe

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Classical and Quantum Dynamics of Constrained Hamiltonian Systems

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Preface

Since the pioneering work of Bergmann in 1949, it has been understood that there exists a direct connection between local symmetries of a Lagrangian and the existence of constraints contained in the Euler-Lagrange equations of motion. It was Dirac who subsequently discussed the Hamiltonian dynamics of constrained systems in a systematic way. Since then numerous papers, with applications to a wide range of problems, have been written on this subject. It is remarkable that even after more than half a century, this subject still raises questions to be answered.

This book is the result of lectures delivered by the authors at the University of Heidelberg. It is intended as an introduction to this interesting field of research, starting from the early work of Dirac up to more recent work, such as the Field-Antifield formalism of Batalin and Vilkovisky. Also included is a brief discussion of gauge anomalies within this formalism. Here we have restricted ourselves to the essentials, and have illustrated the ideas in terms of a simple example, the chiral Schwinger model.

Great emphasis is placed on discussing the subject of constrained Hamiltonian systems in a transparent way without becoming too technical. Proofs, as well as examples, are discussed in great detail, at the risc of being sometimes pedantic. Each example has been chosen to illustrate some important point. The book should thus hopefully provide a good basis for master degree and Ph.D students.

The book is divided into two parts: part one, involving chapters 1 through 8, deals with the classical dynamics of constrained Hamiltonian systems, while the second part is devoted to their quantization. Here it is assumed that the reader is familiar with the Feynman path integral approach to quantization.

Notation

The Einstein summation convention is used for repeated upper and lower indices. Except for Lorentz space-time indices, we have placed indices, labeling the degrees of freedom, at our convenience, so as to keep the expressions most transparent. We have followed to a large extent the notation given below, although in particular cases - such as in examples - we have found it convenient to follow a simpler notation.

Notation

ϕ_α: primary constraints
$\phi^{(1)}_{\alpha_1}$: first class primary constraints
$\phi^{(2)}_{\alpha_2}$: second class primary constraints
φ_a: secondary constraints
$\varphi^{(1)}_{a_1}$: first class secondary constraints
$\varphi^{(2)}_{a_2}$: second class secondary constraints
$\Omega^{(1)}_{A_1}$: complete set of first class constraints
$\Omega^{(2)}_{A_2}$: complete set of second class constraints
Γ_P: subspace defined by primary constraints
Γ: subspace defined by all constraints
Γ^*: subspace defined by the constraints and gauge conditions
H_0: canonical Hamiltonian on the primary constrained surface Γ_P
H: Hamiltonian weakly equivalent to H_0
$\tilde{H}$: BFT embedded Hamiltonian
H_E: extended Hamiltonian
$\tilde{H}_E$ fully extended Hamiltonian, including the gauge conditions
H^*: Hamiltonian on the reduced phase-space
$Q_{\mathcal{B}}$: generator of BRST transformations: $\delta_{\mathcal{B}}\mathcal{F} = \epsilon\{Q_{\mathcal{B}}, \mathcal{F}\} = \{\mathcal{F}, Q_{\mathcal{B}}\}\epsilon$
s: operator inducing "left" BRST transformations: $s\mathcal{F} = \{Q_{\mathcal{B}}, \mathcal{F}\}$
$\hat{s}$: operator inducing "right" BRST transformations $\hat{s}\mathcal{F} = \{\mathcal{F}, Q_{\mathcal{B}}\}$
$\overset{\vee}{s}$: operator inducing "right" BRST transformations on Lagrangian level

Contents

Chapter 1

Introduction

Elementary particle physicists are well familiar with Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD), the theories describing the electromagnetic and strong interactions. These theories are based on Lagrangians possessing local (gauge) symmetries. The weak interactions are also based on a gauge theory, but whose gauge invariance is spontaneously broken. In fact, all theories of the fundamental interactions are gauge theories. This makes the study of systems possessing local symmetries so important. As is well known, the quantization of such systems is endowed with special problems, since their classical action is invariant along gauge orbits. On the level of the equations of motion this implies the existence of an infinite number of solutions connected by local (gauge) transformations, which are physically completely equivalent. Several methods of quantization, such as the canonical Gupta-Bleuler formalism, or the functional path integral approach, which makes use of the so-called Faddeev-Popov trick, are well known to theoretical elementary particle physicists. These methods are described in any textbook on Quantum Field Theory.

The question then arises whether there exists a fundamental feature which is common to all gauge theories, and which inhibits their straightforward quantization. The answer is “yes”; they all belong to the so-called class of singular systems, characterized by the property, that the Euler-Lagrange equations of motion include equations which represent constraints among the coordinates and the velocities, and which on Hamiltonian level manifest themselves as so-called “first class” constraints among canonical variables. These constraints differ from those familiar from mechanics in that they are part of the equations of motion, and are not imposed from the outside. Although gauge theories are certainly the most important examples of singular Lagrangian systems, they are not the only ones belonging to this class. Even the dynamics of the free Dirac field is that of a constrained system.

A deeper understanding of the quantization of gauge theories requires one to go back to the roots of the quantization problems associated with singular Lagrangians. The basis for understanding such systems was laid down by Bergmann [Bergmann 1949, Anderson 1951], and by Dirac [Dirac 1950/1964]. Since then an enormous amount of papers have been written about constrained Hamiltonian systems.

This book is mainly devoted to the study of the classical and quantum dynamics of theories where the Lagrangian exhibits local symmetries. Its purpose is to introduce the reader to this interesting field of constrained dynamical systems, and to provide him with the tools for quantizing such systems. This will require first of all an extensive discussion of the classical Hamiltonian formulation of singular systems. Our language will be kept as simple as possible, and the concepts we shall introduce will always be illustrated by examples in a transparent way as possible. The classical discussion is contained in chapters 2 through 8. The remaining chapters are devoted to the quantization of singular systems.

The book is organized as follows. In chapter 2 we first discuss what is meant by a singular Lagrangian system, and show that systems with a local (gauge) symmetry fall into this class. Local symmetries of the action are not always easily detected. It is however crucial to unravel them, since their knowledge is required for the quantization of such systems. Although their quantization is carried out on Hamiltonian level, it is instructive to first present a systematic method for detecting local symmetries on Lagrangian level. This is the main objective of chapter 2, where we shall show that singular systems lead to so-called "gauge identities", from which the coordinate transformations, leaving the action invariant, can be extracted.

In chapter 3 we then proceed to the classical Hamiltonian formulation based on the pioneering work of Dirac. It is within this framework that Dirac has provided a systematic classification of the constraints into *primary*, *secondary*, *first* and *second class*, where the first class constraints play a central role as generators of the gauge symmetries.

Chapter 4 is devoted to an alternative method for handling singular systems: the symplectic approach. This is a first order Lagrangian approach to singular systems, and we shall study in detail the equations of motion and their solution. The symplectic approach has been used by Faddeev and Jackiw [Faddeev 1988] to present a new method for handling constrained systems, which, in principle, avoids the concepts of primary constraints (which have no analog on the Lagrangian level), and in fact dispenses of the entire classification of the constraints into primary, secondary, first and second class. This method is however of limited applicability.

The Hamiltonian approach for detecting local symmetries is the phase-space counterpart to our discussion in chapter 2, and is the subject of chapter 5. We shall elucidate the problem from several points of view and are thereby led to the so-called "Dirac conjecture", which has been the subject of numerous papers.

The Dirac conjecture will be discussed in detail in chapter 6, where we make use of the Lagrangian methods discussed chapter 2 to study the local symmetries of the so-called total action, whose variation yields the Hamilton equations of motion proposed by Dirac. This chapter clarifies in which sense the Dirac conjecture can be shown to hold. Interpreted appropriately, this conjecture is shown to hold for various examples cited in the literature, including those which have been claimed to be counterexamples to Dirac's conjecture.

Chapter 7 deals with second class systems whose classical dynamics is beautifully described in terms of Dirac brackets, but whose quantization may pose special problems. For this reason it is useful to embed such systems into a gauge theory, by introducing extra degrees of freedom. The embedded theories can then be quantized using the formalism described in chapter 11. This embedding program, whose systematic implementation was given by Batalin, Fradkin and Tyutin, [Batalin 1987/1991] is the central subject of this chapter. We shall present several examples, and in particular use this formalism to demonstrate the equivalence of the "self dual model" in three space-time dimensions, with the Maxwell-Chern-Simons theory within its gauge invariant sector.

Chapter 8 deals with the Hamilton-Jacobi formulation of the classical dynamics of constrained systems. While this formulation turns out to be straightforward for so-called first class systems, this is not the case for systems involving second class constraints. We shall discuss the reason for this, and present methods for circumventing the barriers imposed by the second class constraints. In particular, the embedding procedure described in chapter 7 will allow one to arrive at a Hamilton-Jacobi like formulation, as we shall demonstrate for several examples.

In the following chapters we then turn to the problem of quantizing systems with first and second class constraints. Chapter 9 will deal with purely second class systems which allow an operator quantization, based on equations of motion formulated using the notion of an extended Hamiltonian. Second class systems also include the case of gauge theories in a non-dynamical gauge.

In chapter 10 we then discuss in detail the quantization of second class systems using functional methods, and point out the difficulties one may be confronted with. These difficulties may be circumvented by first embedding the system into a gauge theory, as was described in chapter 7.

The functional quantization of first class systems in dynamical gauges is the subject of chapter 11 and leads to the Hamiltonian formulation of Batalin, Fradkin and Vilkovisky [Fradkin 1975/77, Batalin 1977], based on the so-called BRST (Becchi, Rouet, Stora, Tyutin) invariance of the Lagrangian action. We shall not follow the axiomatic approach of these authors, but shall motivate the BFV formalism by studying systems whose functional configuration space formulation, obtained using the Faddeev-Popov trick, is well established. We then obtain the corresponding phase-space representation of the partition function and cast it into the form given in [Batalin 1977]. The so-called BRST charge will play a central role as generator of a BRST symmetry. The freedom in the choice of gauge will be concentrated in a BRST exact term, and the BRST invariance of the partition function is guaranteed by the Fradkin-Vilkovisky principal theorem [Fradkin 1977, Batalin 1977]. Important ingredients in this formulation is the closure of the algebra of the first class constraints, as well as of these constraints with the Hamiltonian.

In the Hamiltonian formulation it is not obvious how a particular covariant gauge is to be implemented via a BRST exact contribution to the action. A corresponding configuration space formulation is therefore desirable. This is the subject of the last chapter and leads to the so-called "field-antifield" formulation of Batalin and Vilkovisky [Batalin 1981]. This quantization scheme has been presented by these authors in an axiomatic way. We shall actually derive their formulation for a restricted class of systems, starting from the well established phase-space formulation of Batalin, Fradkin and Vilkovisky. Within the field-antifield formulation the number of degrees of freedom is doubled, and the field-antifield action is the solution of the so-called "quantum master equation". We will discuss only the simplest non-trivial case of irreducible first rank theories. These include the case of $SU(N)$ Yang-Mills theories.

We conclude chapter 12 with a brief discussion of how anomalous gauge theories - i.e., theories whose classical gauge invariance is broken by quantum effects due to the non-invariance of the functional integration measure - may be included in the field-antifield formalism. This subtle subject is outside the scope of this book. We shall therefore limit ourselves to illustrate the role played by the quantum master equation for the case of the chiral Schwinger model, which is anomalous and has been studied in great detail in the literature.

The book includes several appendices. In Appendix A we review some well known facts regarding the Ward and Bianchi identities following from the existence of a local symmetry of the action. This appendix is based on the pioneering work of P. G. Bergmann [Bergmann 1949], who has pointed out the connection between the existence of a local symmetry and a so-called "singular Lagrangian". The work of Bergmann has been reviewed in [Costa 1988]. We shall follow closely the presentation given in this reference.

Appendices B and C deal with technical details concerning the construction of the BRST charge and Hamiltonian for first rank theories.

Appendix D contains a proof of the Fradkin-Vilkovisky Principal Theorem, which provides the basis for the Batalin-Vilkovisky configuration space partition function in chapter 12.

Finally, Appendix E discusses the Fradkin-Vilkovisky path integral representation of the $SU(3)$ Yang-Mills partition function in the α-gauges.

Chapter 2

Singular Lagrangians and Local Symmetries

2.1 Introduction

Symmetries observed in nature play a central role in the construction of any theory describing physical phenomena. If the dynamics is described by a Lagrangian, then this Lagrangian must reflect such symmetries. Not all symmetries of an action can however be observed in nature. This is for example the case for the gauge symmetries of QED or QCD, where observables must be invariant under gauge transformations. Gauge theories fall under the class of constrained systems whose dynamics is derived from a so-called singular Lagrangian. Such theories are of particular interest, since all interactions in nature are believed to be gauge theories, or spontaneously broken versions thereof. As is well known, their quantization poses special problems.

Whether a Lagrangian is singular or not can in general be easily checked by looking for possible zero modes of the Hessian (to be defined below). The connection between a singular Hessian and local symmetries of a Lagrangian has been observed already a long time ago [Bergmann 1949]. Such symmetries lead to so-called Bianchi identities from which this connection can be deduced. The interested reader may consult Appendix A, where we have summarized some well known concepts about local symmetries, as well as consequences following from them. In this chapter we take the opposite point of view and deduce the local symmetries of a Lagrangian whose Hessian matrix is not invertible. In fact, unravelling these symmetries may not be an easy matter. But their explicit knowledge is important for formulating the corresponding quantum theory. It is therefore of great interest to have a systematic way for revealing them. In the following we will present a purely Lagrangian approach to this important problem [Sudarshan 1974]. This will allow the reader to appreciate better the elegant Hamiltonian approach to be discussed in chapter 3, where we show that such symmetries are intimately related to the existence of so-called "first class constraints". Both approaches have their merits. But the connection between the existence of local symmetries, and the existence of first class constraints as generators of these symmetries, is only manifest on Hamiltonian level.

On the Lagrangian level there exist several algorithms [Sudarshan 1974, Chaichian 1994, Shirzad 1998] for detecting the gauge symmetries of a Lagrangian. Unlike the Hamiltonian approach, they have the merit of generating directly the transformation laws in configuration space in terms of an independent set of functions parametrizing the symmetry transformations. Every one of these parameters is directly related to a so-called "gauge identity", and their number equals the number of independent gauge identities.

2.2 Singular Lagrangians

Consider a system with a finite number of degrees of freedom, whose dynamics is governed by a Lagrangian $L(q, \dot{q})$, where q and $\dot{q}$ stand collectively for the coordinates q_i and velocities $\dot{q}_i$ $(i = 1, \cdots, N)$. The associated Euler-Lagrange equations read,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0\,. \tag{2.1}$$

From here it follows that

$$\frac{\partial^2 L}{\partial q_j \partial \dot{q}_i}\frac{dq_j}{dt} + \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i}\frac{d\dot{q}_j}{dt} - \frac{\partial L}{\partial q_i} = 0\,,$$

or

$$\sum_j W_{ij}(q, \dot{q})\ddot{q}_j = \frac{\partial L}{\partial q_i} - \sum_j \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\dot{q}_j\,,$$

where the matrix W is the "Hessian" defined by

$$W_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\,. \tag{2.2}$$

If $det\ W \neq 0$, we can solve the above equation for the accelerations in terms of $\dot{q}_i$ and q_i. On the other hand, if $det\ W = 0$, then W is not invertible. In this case the Hamiltonian equations of motion will not take the standard form, and one speaks of a singular Lagrangian system.

The following two examples illustrate some features which we shall subsequently encounter in our more general discussion of local symmetries.

i) Example 1

Consider the Lagrangian

$$L(q,\dot{q}) = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x-y)^2 \tag{2.3}$$

and the associated action

$$S[q] = \int_{t_1}^{t_2} dt\ L(q(t),\dot{q}(t)) \ , \tag{2.4}$$

where $q = (x,y)$. This Lagrangian is singular in the above sense, since it does not involve $\dot{y}$. Hence, we claim, it will possess a local symmetry. In this example this symmetry can be found by inspection.

Consider a variation $\delta x = \epsilon_1(t)$, $\delta y = \epsilon_2(t)$. Under this variation the change induced in the Lagrangian can be written in the form

$$\delta L = (\dot{x} + y - x)(\dot{\epsilon}_1 - \epsilon_1 + \epsilon_2) + \frac{d}{dt}(x\epsilon_1) \ .$$

Hence the corresponding action (2.4) is left invariant under this transformation if

$$\epsilon_2 = \epsilon_1 - \dot{\epsilon}_1 \ , \tag{2.5}$$

with $\epsilon_1(t)$ vanishing for $t = t_1$ and $t = t_2$. Thus for an arbitrary choice of $\epsilon_1(t)$, subject to this condition, ϵ_2 is given in terms of ϵ_1 and its *first* derivative.

ii) Example 2

Consider next the Lagrangian [Henneaux 1990a]

$$L(q,\dot{q}) = \frac{1}{2}(\dot{q}_2 - e^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2 \ . \tag{2.6}$$

Again this Lagrangian is singular and therefore - we claim - will exhibit a local symmetry. Thus consider a variation $\delta q_i = \epsilon_i(t)$. The corresponding change induced in the Lagrangian is given by

$$\delta L = (\dot{q}_2 - e^{q_1})(\dot{\epsilon}_2 - e^{q_1}\epsilon_1) + (\dot{q}_3 - q_2)(\dot{\epsilon}_3 - \epsilon_2) \ .$$

Hence the Lagrangian (2.6) is invariant if

$$\begin{aligned} \epsilon_3(t) &= \alpha(t) \ , \\ \epsilon_2(t) &= \frac{d}{dt}\alpha(t) \ , \end{aligned}$$

$$\epsilon_1(t) = e^{-q_1(t)} \frac{d^2}{dt^2} \alpha(t) \ , \tag{2.7}$$

where $\alpha(t)$ is an arbitrary function of t. Notice that, as in our previous example, the symmetry transformation only involves one arbitrary function. Furthermore, in this example, the variation δq_1 not only involves the arbitrary function $\alpha(t)$, but also depends implicitly on t via the coordinate q_1.

In the following section we present a general algorithm which allows one to determine the variations δq_i which leave an action invariant.

2.3 Algorithm for detecting local symmetries on Lagrangian level

On the Lagrangian level there exists a well-known algorithm [Sudarshan 1974] for detecting all gauge symmetries of a Lagrangian.

Consider an infinitesimal variation of the coordinates and velocities

$$q_i(t) \to q_i(t) + \delta q_i(t) \ ,$$

$$\dot{q}_i(t) \to \dot{q}_i(t) + \frac{d}{dt} \delta q_i(t) \ ,$$

where $i = 1, \cdots, N$, and use has been made of the property that $\delta \dot{q}_i = \frac{d}{dt} \delta q_i$. This transformation is a local symmetry of the Euler-Lagrange equations of motion if the action (2.4) is invariant under this transformation. This is the case if the variation of the Lagrangian is a total derivative, i.e., $\delta L = \frac{d}{dt} \delta F$, with δF vanishing at the upper and lower limit of integration. In computing the variation no use is made of the equations of motion.

Consider now a Lagrangian $L(q, \dot{q})$. Under the above infinitesimal transformation the change in the action (2.4) is given by

$$\delta S = -\int_{t_1}^{t_2} dt \sum_i E_i^{(0)}(q, \dot{q}, \ddot{q}) \delta q_i \ , \tag{2.8}$$

where $E_i^{(0)}$ is the Euler derivative

$$E_i^{(0)}(q, \dot{q}, \ddot{q}) = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} \tag{2.9}$$

and $\delta q_i(t_1) = \delta q_i(t_2) = 0$. On shell we have that

$$E_i^{(0)} = 0 \, , \quad i = 1, \cdots, N \quad (\textit{on shell}) \ . \tag{2.10}$$

Our aim is to determine the variations δq_i for which δS vanishes identically, i.e.,

$$\int_{t_1}^{t_2} dt \ \sum_i E_i^{(0)}(q, \dot{q}, \ddot{q})\delta q_i \equiv 0 \ . \tag{2.11}$$

In general, $E_i^{(0)}$ is of the form

$$E_i^{(0)} = \sum_j W_{ij}^{(0)}(q, \dot{q})\ddot{q}_j + K_i^{(0)}(q, \dot{q}) \ , \tag{2.12}$$

where $W_{ij}^{(0)}$ is the Hessian (2.2). For reasons which will become clear below, we have supplied the terms in (2.12) with a superscript "0" standing for "zeroth level".

Now, as we have just seen, for singular Lagrangians the Lagrange equations of motion cannot be solved for all of the accelerations. If R_W is the rank of the Hessian, then there exist $N - R_W$ constraints in the theory, which on Lagrangian level present themselves as relations between coordinates q_i and velocities $\dot{q}_i$. Thus for an $N \times N$ Hessian $W_{ij}^{(0)}$ of rank R_W, there exist $N - R_W$ independent left (or right), zero-mode eigenvectors $\vec{w}^{(0;k)}$ satisfying

$$\sum_{i=1}^{N} w_i^{(0;k)}(q, \dot{q}) W_{ij}^{(0)}(q, \dot{q}) = 0 \ ; \quad k = 1, \cdots, N - R_W \ .$$

It then follows from (2.12) that the functions defined by

$$\Phi^{(0;k)} := \sum_{i=1}^{N} w_i^{(0;k)}(q, \dot{q}) E_i^{(0)}(q, \dot{q}, \ddot{q})$$

only depend on the coordinates and velocities, and vanish on the subspace of physical trajectories, i.e.,

$$\Phi^{(0;k)}(q, \dot{q}) = 0 \ ; \quad k = 1, \cdots, N - R_W \quad (\textit{on shell}) \ .$$

We refer to them as the zero-generation constraints. [1] Not all of the $\Phi^{(0;k)}$ may, however, be linearly independent. In this case one can find linear combinations of the zero mode eigenvectors, $v_i^{(0;n_0)} = \sum_k c_k^{(n_0)} w_i^{(0;k)}$, such that we have identically

$$G^{(0;n_0)} := \vec{v}^{(0;n_0)} \cdot \vec{E}^{(0)} \equiv 0 \ ; \quad n_0 = 1, \cdots, N_0 \ . \tag{2.13}$$

We call them "gauge identities" for reasons which will become clear soon. These gauge identities also hold off shell. As an immediate consequence we have that

[1] For simplicity we shall suppress in the following the explicit dependence of the constraints and zero modes on q and $\dot{q}$.

any variation of the form $\delta q_i = \sum_{n_0} \epsilon_{n_0}(t) v_i^{(0;n_0)}$ will leave the action invariant (cf. (2.11)). The remaining "zero-generation" zero modes, which we denote by $\vec{u}^{(0;\bar{n}_0)}$, then lead to genuine constraints, depending only on the coordinates and velocities, and vanishing on shell,

$$\phi^{(0;\bar{n}_0)} = 0 \quad (\textit{on shell}) \ ,$$

where

$$\phi^{(0;\bar{n}_0)} = \vec{u}^{(0;\bar{n}_0)} \cdot \vec{E}^{(0)} \ ; \quad \bar{n}_0 = 1, \cdots, \bar{N}_0 \ . \tag{2.14}$$

The algorithm now proceeds as follows. We separate the gauge identities (2.13) from the non-trivial constraints (2.14), and list them separately in a table, since they will be of direct relevance for determining the local symmetry transformations. Next we look for possible additional constraints by searching for further functions of the coordinates and velocities which vanish on the subspace of physical trajectories. To this effect consider the following $N + \bar{N}_0$ component vector constructed from $\vec{E}^{(0)}$ and the time derivative of the constraints (2.14):

$$\left(E_{i_1}^{(1)}\right) := \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}(\vec{u}^{(0;1)} \cdot \vec{E}^{(0)}) \\ \cdot \\ \cdot \\ \cdot \\ \frac{d}{dt}(\vec{u}^{(0;\bar{N}_0)} \cdot \vec{E}^{(0)}) \end{pmatrix} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\vec{\phi}^{(0)} \end{pmatrix} \ , \tag{2.15}$$

where $\vec{\phi}^{(0)}$ is an $\bar{N}_0$ component column vector with components $\phi^{(0;\bar{n}_0)}$, ($\bar{n}_0 = 1, \cdots, \bar{N}_0$). Since the constraints $\phi^{(0;\bar{n}_0)} = 0$ hold for all times, it follows that $\vec{E}^{(1)} = 0$ on shell.

Because the constraints are only functions of q and $\dot{q}$, the components $E_{i_1}^{(1)}$ can again be written in a form paralleling (2.12),

$$E_{i_1}^{(1)} = \sum_{j=1}^{N} W_{i_1 j}^{(1)}(q,\dot{q})\ddot{q}_j + K_{i_1}^{(1)}(q,\dot{q}) \ . \tag{2.16}$$

Here $\mathbf{W}^{(1)}$ is now the "level 1" $(N + \bar{N}_0) \times N$ matrix

$$\left(W_{i_1 i}^{(1)}\right) := \begin{pmatrix} \mathbf{W}^{(0)} \\ \vec{\nabla}_{\dot{q}}(\vec{u}^{(0;1)} \cdot \vec{E}^{(0)}) \\ \cdot \\ \cdot \\ \cdot \\ \vec{\nabla}_{\dot{q}}(\vec{u}^{(0;\bar{N}_0)} \cdot \vec{E}^{(0)}) \end{pmatrix} \ ,$$

where $\vec{\nabla}_{\dot{q}}$ is a *row* vector with components labeled by i $(i = 1, \cdots, N)$. The $(N + \bar{N}_0)$ component vector $\vec{K}^{(1)}$ is given by

$$\left(K_{i_1}^{(1)}\right) := \begin{pmatrix} \vec{K}^{(0)} \\ \frac{\partial}{\partial q_j}(\vec{u}^{(0;1)} \cdot \vec{E}^{(0)})\dot{q}_j \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial}{\partial q_j}(\vec{u}^{(0;\bar{N}_0)} \cdot \vec{E}^{(0)})\dot{q}_j \end{pmatrix} ,$$

where a summation over j is understood.

We next look again for (left) zero-modes of $\mathbf{W}^{(1)}$. These zero modes include of course those of the previous level, augmented by an appropriate number of zeroes. They just reproduce the previous constraints and are therefore not considered. The remaining zero modes (if they exist), when contracted with $\vec{E}^{(1)}$, lead to expressions at "level 1" which can either be written as a linear combination of the previous constraints (we label them by n_1), in which case we are led to new gauge identities at level 1,

$$G^{(1;n_1)} := \vec{v}^{(1;n_1)} \cdot \vec{E}^{(1)} - \sum_{\bar{n}_0=1}^{\bar{N}_0} M_{n_1,\bar{n}_0}^{(1,0)}(\vec{u}^{(0;\bar{n}_0)} \cdot \vec{E}^{(0)}) \equiv 0 \ ; \quad n_1 = 1, \cdots, N_1 \ , \tag{2.17}$$

or else, represent genuine new constraints at level 1:

$$\phi^{(1;\bar{n}_1)} := \vec{u}^{(1;\bar{n}_1)} \cdot \vec{E}^{(1)} = 0 \ ; \quad \bar{n}_1 = 1, \cdots, \bar{N}_1 \ \ (\textit{on shell}). \tag{2.18}$$

We now adjoin the new gauge identities (2.17) to the previous identities in our table. With the remaining new constraints (2.18) we proceed as before, adjoining their time derivative to (2.15), and construct $W_{i_2 i}^{(2)}$ as well as $K_{i_2}^{(2)}$. The iterative process will terminate at some level M if either *i*) there are no further zero modes, or *ii*) the constraints generated are linear combinations of the previous ones, and hence lead to gauge identities only. At this point our algorithm has unravelled all the constraints of the Euler-Lagrange equations of motion.

We now look for the maximal set of *linearly independent* gauge identities generated by the algorithm. At each level ℓ, the elements of this set are of the form

$$G^{(0,n_0)} := \vec{v}^{(0;n_0)} \cdot \vec{E}^{(0)} \equiv 0 \ , \tag{2.19}$$

$$G^{(\ell,n_\ell)} := \vec{v}^{(\ell;n_\ell)} \cdot \vec{E}^{(\ell)} - \sum_{\ell'=0}^{\ell-1} \sum_{\bar{n}_{\ell'}=1}^{\bar{N}_{\ell'}} M_{n_\ell,\bar{n}_{\ell'}}^{(\ell,\ell')} \phi^{(\ell';\bar{n}_{\ell'})} \equiv 0 \, , \tag{2.20}$$

where $\ell = 1, \cdots, N_\ell$, and where the $M^{(\ell,\ell')}_{n_\ell,n_{\ell'}}$ are only functions of q and $\dot{q}$, and

$$\phi^{(\ell;\bar{n}_\ell)} = \vec{u}^{(\ell;\bar{n}_\ell)} \cdot \vec{E}^{(\ell)} \ ; \quad (\bar{n}_\ell = 1, \cdots, \bar{N}_\ell) \tag{2.21}$$

are the genuine constraints (depending only on q and $\dot{q}$) generated by the algorithm at level ℓ. On the other hand, $\vec{E}^{(\ell)}$ is given by the $N + \bar{N}_0 + \cdots + \bar{N}_{\ell-1}$ component vector

$$\vec{E}^{(\ell)} := \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\vec{\phi}^{(0)} \\ \cdot \\ \cdot \\ \cdot \\ \frac{d}{dt}\vec{\phi}^{(\ell-1)} \end{pmatrix} , \tag{2.22}$$

where $\vec{\phi}^{(\ell)}$ is a column vector with $\bar{N}_\ell$ components $\phi^{(\ell;\bar{n}_\ell)}$. Because of (2.21) and (2.22), we can proceed iteratively to express each of the constraints $\phi^{(\ell;\bar{n}_\ell)}$ in terms of $\phi^{(0;\bar{n}_0)} = \vec{u}^{(0;\bar{n}_0)} \cdot \vec{E}^{(0)}$ and time derivatives thereof. One then readily verifies that these constraints are of the form

$$\phi^{(\ell;\bar{n}_\ell)} = \sum_i \sum_{m=0}^{\ell} \eta^{(\ell;\bar{n}_\ell)}_{mi} \frac{d^m}{dt^m} E^{(0)}_i = 0 \, .$$

From here, and the expression (2.22), we conclude that the identities (2.19) are of the following general form

$$G^{(\ell;n_\ell)} = \sum_i \sum_{m=0}^{\ell} \rho^{(\ell;n_\ell)}_{mi} \frac{d^m}{dt^m} E^{(0)}_i \equiv 0 \ ,$$

where $\rho^{(\ell;n_\ell)}_{mi}$ are in general functions of the coordinates and time derivatives thereof. It therefore also follows that

$$\sum_{\ell=0}^{M} \sum_{n_\ell=1}^{N_\ell} \epsilon^{(\ell;n_\ell)}(t) G^{(\ell;n_\ell)} \equiv 0 \ ,$$

where the $\epsilon^{(\ell;n_\ell)}(t)$ are arbitrary functions of time. This identity can be written in the form

$$\sum_i \delta q_i E^{(0)}_i - \frac{d}{dt} F \equiv 0 \ , \tag{2.23}$$

where

$$\delta q_i = \sum_{\ell=0}^{M} \sum_{n_\ell=1}^{N_\ell} \sum_{m=0}^{\ell} (-1)^m \frac{d^m}{dt^m} \left(\rho^{(\ell;n_\ell)}_{mi} \epsilon^{(\ell;n_\ell)}(t) \right) \ , \tag{2.24}$$

and F is a function of the coordinates and their time derivatives, as well as a linear function of the parameters and their derivatives up to order $M-1$, where $\ell = M$ is the highest level. Note that the arbitrary functions $\epsilon^{(\ell;n_\ell)}(t)$ are labeled by ℓ, and n_ℓ. Hence the number of independent functions equals the number of gauge identities generated by the algorithm. Let us collect the indices ℓ and n_ℓ into a single index a. Then (2.24) reads

$$\delta q_i = \sum_{m,a} 1)^m \frac{d^m}{dt^m}\left(\rho_{mi}^{(a)}\epsilon^{(a)}(t)\right) . \tag{2.25}$$

Now, integrating (2.23) in time from t_1 to t_2, and using the fact that the total derivative will not contribute if the $\epsilon^{(\ell;\bar{n}_\ell)}(t)$ and their derivatives vanish for $t = t_1$ and $t = t_2$ (implying the vanishing of δq_i at the endpoints in (2.11)), we conclude that the variation (2.25) is a symmetry of the action, and hence of the Lagrange equations of motion. It should be noted, however, that a term of the form

$$\Delta q_i = \sum_j a_{ij} E_j^{(0)} ,$$

with an arbitrary antisymmetric tensor a_{ij}, can always be added to (2.25), and it will remain a symmetry transformation, since $E_i^{(0)}\Delta q_i \equiv 0$. These variations, which vanish on shell, are called "trivial gauge transformations". They have no physical significance.

In the following we consider some examples which illustrate the above iterative methods for generating the gauge identities. Since in the first two examples there exists only one zero mode of the type u or v at each level, we will omit the indexes n_ℓ and $\bar{n}_\ell$ labeling such zero modes at level ℓ.

2.4 Examples

Example 1

Consider once more the Lagrangian (2.3). Let $x_1 = x$ and $x_2 = y$. Then

$$E_i^{(0)} = \sum_j W_{ij}^{(0)}\ddot{x}_j + K_i^{(0)} ,$$

where

$$\mathbf{W}^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\mathbf{K}^{(0)} = \begin{pmatrix} \dot{x}_2 + x_2 - x_1 \\ x_1 - x_2 - \dot{x}_1 \end{pmatrix} .$$

There exists only a single left zero mode, $\vec{u}^{(0)} = (0,1)$. Its contraction with $\vec{E}^{(0)}$ does not vanish identically, but generates a constraint:

$$\phi^{(0)} \equiv \vec{u}^{(0)} \cdot \vec{E}^{(0)} = x_1 - x_2 - \dot{x}_1 = 0 \quad (\textit{on shell}) \ . \tag{2.26}$$

Hence at level zero we have one constraint and no gauge identity. We next construct $\vec{E}^{(1)}$:

$$\vec{E}^{(1)} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\left(\vec{u}^{(0)} \cdot \vec{E}^{(0)}\right) \end{pmatrix} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt} E_2^{(0)} \end{pmatrix} \ , \tag{2.27}$$

or, written out explicitly

$$\vec{E}^{(1)} = \begin{pmatrix} \ddot{x}_1 + \dot{x}_2 - x_1 + x_2 \\ x_1 - x_2 - \dot{x}_1 \\ \dot{x}_1 - \dot{x}_2 - \ddot{x}_1 \end{pmatrix} \ .$$

$\vec{E}^{(1)}$ can again be written in the form

$$E_{i_1}^{(1)} = \sum_j W_{i_1 j}^{(1)} \ddot{x}_j + K_{i_1}^{(1)} \ ,$$

where the 3×2 matrix $\mathbf{W}^{(1)}$ is given by

$$\mathbf{W}^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{pmatrix} \ ,$$

and

$$\mathbf{K}^{(1)} = \begin{pmatrix} \dot{x}_2 + x_2 - x_1 \\ x_1 - x_2 - \dot{x}_1 \\ \dot{x}_1 - \dot{x}_2 \end{pmatrix} \ .$$

The left zero modes of $\mathbf{W}^{(1)}$ are given by $(0,1,0)$ and $(1,0,1)$. The first zero mode is just the previous zero mode augmented by one zero, and reproduces the previous constraint. We need not consider it again. The second zero mode reproduces the (negative) of the constraint (2.26), and is thus of the type v:

$$\vec{v}^{(1)} \cdot \vec{E}^{(1)} \equiv -\vec{u}^{(0)} \cdot \vec{E}^{(0)} \ ,$$

where $\vec{v}^{(1)} = (1,0,1)$. This leads to the gauge identity

$$G^{(1)} = \vec{v}^{(1)} \cdot \vec{E}^{(1)} + \vec{u}^{(0)} \cdot \vec{E}^{(0)} \equiv 0$$

at level 1. Noting from (2.27) that

$$\vec{v}^{(1)} \cdot \vec{E}^{(1)} = E_1^{(0)} + \frac{d}{dt}E_2^{(0)} \ ,$$

the above identity takes the form

$$E_1^{(0)} + \frac{d}{dt}E_2^{(0)} + E_2^{(0)} \equiv 0 \ .$$

Multiplying this expression by an arbitrary function of time, $\epsilon(t)$, we are led to

$$\epsilon E_1^{(0)} - (\dot{\epsilon} - \epsilon)E_2^{(0)} = -\frac{d}{dt}(\epsilon E_2^{(0)}) \ .$$

This is of the form

$$\sum_i E_i^{(0)} \delta x_i = \frac{dF}{dt}$$

with

$$\delta x_1 = \epsilon\,, \quad \delta x_2 = \epsilon - \dot{\epsilon} \ ,$$

which agrees with (2.5). This variation leaves the action invariant if $\epsilon(t_i) = 0$ $(i = 1, 2)$.

Example 2

Consider the Lagrangian (2.6). The corresponding Euler derivative has the form (2.12), with

$$W^{(0)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ ,$$

and

$$\vec{K}^{(0)} = \begin{pmatrix} \dot{q}_2 e^{q_1} - e^{2q_1} \\ \dot{q}_3 - q_2 - \dot{q}_1 e^{q_1} \\ -\dot{q}_2 \end{pmatrix} \ .$$

$W^{(0)}$ possesses one left null-eigenvector $u^{(0)} = (1, 0, 0)$, implying a level zero constraint:

$$\phi^{(0)} \equiv \vec{u}^{(0)} \cdot \vec{E}^{(0)} = e^{q_1}(\dot{q}_2 - e^{q_1}) = 0 \quad (\mathit{on\ shell}) \ . \tag{2.28}$$

Next we search for further functions of the coordinates and velocities which vanish on shell, i.e. for $\vec{E}^{(0)} = 0$. To this effect we construct $\vec{E}^{(1)}$ as given by (2.15):

$$\vec{E}^{(1)} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\phi^{(0)} \end{pmatrix} \ . \tag{2.29}$$

One then finds that $W^{(1)}$ and $\vec{K}^{(1)}$ in (2.16) are given by

$$W^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & e^{q_1} & 0 \end{pmatrix} \;,$$

and

$$\vec{K}^{(1)} = \begin{pmatrix} \dot{q}_2 e^{q_1} - e^{2q_1} \\ \dot{q}_3 - q_2 - \dot{q}_1 e^{q_1} \\ -\dot{q}_2 \\ e^{q_1}\dot{q}_1\dot{q}_2 - 2\dot{q}_1 e^{2q_1} \end{pmatrix} \;.$$

A new left-null eigenvector of $W^{(1)}$ at level one is clearly

$$\vec{u}^{(1)} = (0, -e^{q_1}, 0, 1) \;,$$

and leads to a new constraint

$$\vec{u}^{(1)} \cdot \vec{E}^{(1)} = \dot{q}_1 \phi^{(0)} + e^{q_1}(q_2 - \dot{q}_3) = 0 \;. \tag{2.30}$$

Since $\phi^{(0)} = 0$, we may choose instead for the new constraint function

$$\begin{aligned} \phi^{(1)} &= e^{q_1}(q_2 - \dot{q}_3) = \vec{u}^{(1)} \cdot \vec{E}^{(1)} - \dot{q}_1 \vec{u}^{(0)} \cdot \vec{E}^{(0)} \\ &= -e^{q_1} E_2^{(0)} + \frac{d}{dt} E_1^{(0)} - \dot{q}_1 E_1^{(0)} \;, \end{aligned} \tag{2.31}$$

which also vanishes on the space of solutions. Taking its time derivative, and constructing $\vec{E}^{(2)}$, $W^{(2)}$ and $\vec{K}^{(2)}$ one obtains

$$\vec{E}^{(2)} = \begin{pmatrix} \vec{E}^{(1)} \\ \frac{d}{dt}\phi^{(1)} \end{pmatrix} \;, \tag{2.32}$$

$$W^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & e^{q_1} & 0 \\ 0 & 0 & -e^{q_1} \end{pmatrix} \;,$$

and

$$\vec{K}^{(2)} = \begin{pmatrix} \vec{K}^{(1)} \\ e^{q_1}(\dot{q}_1 q_2 - \dot{q}_1 q_3 + \dot{q}_2) \end{pmatrix} \;.$$

The matrix $W^{(2)}$ possesses a new left-zero mode of type v given by $\vec{v}^{(2)} = (0, 0, e^{q_1}, 0, 1)$. A simple calculation shows that this vector generates a gauge identity:

$$\vec{v}^{(2)} \cdot \vec{E}^{(2)} - \dot{q}_1(\vec{u}^{(1)} \cdot \vec{E}^{(1)}) + \dot{q}_1^2(u^{(0)} \cdot \vec{E}^{(0)}) \equiv 0 \;. \tag{2.33}$$

The final step now consists in reducing this expression to a form involving only the Euler derivative $\vec{E}^{(0)}$ and time derivatives thereof. Recalling (2.30) and (2.28), the gauge identity (2.33) can be rewritten as follows

$$e^{q_1}E_3^{(0)} - \frac{d}{dt}\left(e^{q_1}E_2^{(0)} + \dot{q}_1E_1^{(0)} - \frac{d}{dt}E_1^{(0)}\right) + \dot{q}_1\left(e^{q_1}E_2^{(0)} - \frac{d}{dt}E_1^{(0)}\right) + \dot{q}_1^2E_1^{(0)} \equiv 0$$

Multiplying this identity with $\epsilon(t)$ one finds that it can be written in the form (2.23), with

$$F = \left[-e^{q_1}E_2^{(0)} - 2\dot{q}_1E_1^{(0)} + \frac{d}{dt}E_1^{(0)}\right]\epsilon \ ,$$

and

$$\begin{aligned}
\delta q_3 &= e^{q_1}\epsilon \ , \\
\delta q_2 &= e^{q_1}(\dot{\epsilon} + \epsilon\dot{q}_1) \ , \\
\delta q_1 &= \ddot{\epsilon} + 2\dot{q}_1\dot{\epsilon} + \epsilon\ddot{q}_1 + \epsilon\dot{q}_1^2 \ .
\end{aligned}$$

By setting $\epsilon = e^{-q_1}\alpha(t)$ we retrieve the transformation (2.7). The same would have been obtained, if we had replaced the constraint (2.28) by $\tilde{\phi}^{(1)} = \dot{q}_2 - e^{q_1}$, and allowed ϵ to be an explicit function of time only. This shows that we must allow the parameters to depend on time explicitly, as well as implicitly through the coordinates and their higher derivatives in time. As we have seen in our example, this is tied to the way we choose to define the constraints. Such redefinitions evidently presume some regularity conditions, ensuring that such reparametrizations will not imply the introduction of new constraints. We will come back to this point in chapter 6.

Example 3

We now consider an example of a system leading to two gauge identities. The associated symmetry transformations will hence involve two arbitrary functions. [2]

Consider the Lagrangian

$$L = \frac{1}{2}\dot{x}_1^2 + \dot{x}_1(x_2 - x_3) + \frac{1}{2}(x_1 - x_2 + x_3)^2 \ .$$

For the Euler derivative (2.12) one finds that

$$W^{(0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ ,$$

[2] As we will see in chapter 3 this corresponds to the number of "primary" constraints in the Hamiltonian formalism.

and

$$\vec{K}^{(0)} = \begin{pmatrix} \dot{x}_2 - \dot{x}_3 - x_1 + x_2 - x_3 \\ x_1 - x_2 + x_3 - \dot{x}_1 \\ x_2 - x_1 - x_3 + \dot{x}_1 \end{pmatrix} .$$

The matrix $W^{(0)}$ possesses two obvious (zero level) zero modes, which we now label as described in section 2:

$$\vec{u}^{(0,1)} = (0,1,0)$$
$$\vec{u}^{(0,2)} = (0,0,1) .$$

Scalar product multiplication of the zero modes with the Euler derivative yields

$$\vec{u}^{(0.1)} \cdot \vec{E}^{(0)} \equiv \phi^{(0,1)} ,$$
$$\vec{u}^{(0,2)} \cdot \vec{E}^{(0)} \equiv -\phi^{(0,1)} ,$$

where

$$\phi^{(0,1)} = x_1 - x_2 + x_3 - \dot{x}_1 .$$

Hence only one new constraint is generated, and we have one gauge identity,

$$\vec{u}^{(0,1)} \cdot \vec{E}^{(0)} + \vec{u}^{(0,2)} \cdot \vec{E}^{(0)} \equiv 0 ,$$

or

$$E_2^{(0)} + E_3^{(0)} \equiv 0 . \tag{2.34}$$

Proceeding with the Lagrangian algorithm we now construct $E^{(1)}$, $W^{(1)}$ and $\vec{K}^{(1)}$:

$$\vec{E}^{(1)} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\phi^{(0,1)} \end{pmatrix} ,$$

$$W^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} ,$$

and

$$\vec{K}^{(1)} = \begin{pmatrix} \vec{K}^{(0)} \\ \dot{x}_1 - \dot{x}_2 + \dot{x}_3 \end{pmatrix} .$$

$W^{(1)}$ leads to a new zero mode (in addition to those of $W^{(0)}$, augmented by a zero),

$$\vec{u}^{(1,1)} = (1,0,0,1)$$

and to the constraint

$$\phi^{(1,1)} = \vec{u}^{(1,1)} \cdot \vec{E}^{(1)} = E_1^{(1)} + E_4^{(1)}$$
$$= \dot{x}_1 - x_1 + x_2 - x_3 = 0 \ .$$

We now notice that this is actually no new constraint. In fact

$$\phi^{(1,1)} + \phi^{(0,1)} \equiv 0 \ .$$

This leads to a new gauge identity,

$$\vec{u}^{(1,1)} \cdot \vec{E}^{(1)} + \vec{u}_1^{(0,1)} \cdot \vec{E}^{(0)} = E_1^{(1)} + E_4^{(1)} + E_2^{(0)} \equiv 0 \ . \tag{2.35}$$

Multiplying (2.34) and (2.35) by arbitrary functions $\alpha(t)$ and $\beta(t)$, respectively, and noting that $E_1^{(1)} = E_1^{(0)}$ and $E_4^{(1)} = \frac{d}{dt}E_2^{(0)}$ one is then led to the identity

$$\beta E_1^{(0)} + (\beta - \dot{\beta} + \alpha)E_2^{(0)} + \alpha E_3^{(0)} + \frac{d}{dt}(\beta E_2^{(0)}) \equiv 0 \ .$$

Comparing this expression with (2.23), we conclude that the following transformations leave the action invariant:

$$\begin{aligned} \delta x_1 &= \beta, \\ \delta x_2 &= \beta - \dot{\beta} + \alpha, \\ \delta x_3 &= \alpha \ . \end{aligned} \tag{2.36}$$

2.5 Generator of gauge transformations and Noether identities

In the previous section we have related local symmetries of a Lagrangian to gauge (or Noether) identities. Our method allowed us to unravel systematically the local symmetries of a given action. In particular this method showed that in general time derivatives of gauge parameters are involved. The corresponding infinitesimal transformations (2.25) can also be written in the form

$$\delta q_i(t) = \int dt' \ R_a^i(t,t')\epsilon^{(a)}(t') \tag{2.37}$$

where $R_a^i(t,t')$ is a local kernel (generating function) whose explicit form can be read off from (2.25).

We now show that the operators

$$\Gamma_a(t') = -i \int dt \ R_a^i(t,t')\frac{\delta}{\delta q_i(t)}$$

can be regarded as the generators of the infinitesimal gauge transformation (2.25). [3] For this it is convenient to compactify the notation by absorbing the time-dependences into the respective discrete indices $((i,t) \to i;\ (a,t) \to a)$, and omitting the integral-sign, which now becomes part of the summations. In this notation the variation of the action (2.8) becomes

$$\delta S = -E_i^{(0)} R_a^i \epsilon^a \ .$$

With "essential" parameters ϵ^a it follows that for a symmetry transformation (i.e., for $\delta S = 0$) we have the Noether identities [4]

$$E_i^{(0)} R_a^i \equiv 0 \,. \tag{2.38}$$

Similarly, following this notation, the infinitesimal transformations (2.37) can be written in the form

$$\delta q_i = i\epsilon^a \Gamma_a q_i \ ,$$

where

$$\Gamma_a = -iR_a^j \frac{\delta}{\delta q_j} \ .$$

A finite gauge transformation is then given by

$$\vec{q} \to \vec{G}_{(\lambda)}(\vec{q}) = e^{i\lambda^a \Gamma_a} \vec{q} \ .$$

The commutator of two generators can now be easily computed, [5]

$$\begin{aligned}
[\Gamma_a, \Gamma_b] &= -[R_a^i \frac{\delta}{\delta q_i},\ R_b^j \frac{\delta}{\delta q_j}] \\
&= -R_a^i \left(\frac{\delta}{\delta q_i} R_b^j \right) \frac{\delta}{\delta q_j} + R_b^j \left(\frac{\delta}{\delta q_j} R_a^i \right) \frac{\delta}{\delta q_i} \\
&= r_{ab}^i \frac{\delta}{\delta q_i} \ ,
\end{aligned}$$

where

$$r_{ab}^i = \left(R_b^j \frac{\delta}{\delta q_j} R_a^i - R_a^j \frac{\delta}{\delta q_j} R_b^i \right) \ . \tag{2.39}$$

---

[3] We follow here [Gitman 1990].

[4] By "essential parameters" we mean, that the matrix R_a^i possesses no right zero modes. In this case the theory is called "irreducible". This is the case for our systematic construction, where all gauge identities are linearly independent. If the above matrix possesses right zero modes, then the number of Noether identities is reduced. One then distinguishes between "first stage", "second stage", etc. reducible theories (see [Gomis 1995]). Here we will consider only the irreducible case.

[5] We made use of the functional-derivative notation in order to remind the reader that the labels i, a etc. also include a continuous time variable. Thus summations will also include time-integrals.

By differentiating the Noether identities (2.38) with respect to q_j we obtain further useful identities:

$$\frac{\delta^2 S}{\delta q_j \delta q_i} R_a^i + \frac{\delta S}{\delta q_i} \frac{\delta R_a^i}{\delta q_j} \equiv 0 \ . \tag{2.40}$$

From here we see that on the space of physical trajectories ($\frac{\delta S}{\delta q_i} = 0$), $\frac{\delta^2 S}{\delta q_i \delta q_j}$ is degenerate, with R_a^j the right zero modes.

From the identities (2.40) and the definition (2.39), it follows that

$$\frac{\delta S}{\delta q_i} r_{ab}^i = 0 \ ,$$

or

$$E_i^{(0)} r_{ab}^i = 0 \ .$$

Hence (2.39) is also a generating function and will be of the form

$$r_{ab}^i = f_{ab}^c R_c^i + T_{ab}^i \ , \tag{2.41}$$

where

$$T_{ab}^i = a_{ab}^{ij} E_j^{(0)} \ ,$$

with a_{ab}^{ij} an antisymmetric tensor in the upper indices. We thus arrive at the algebra

$$[\Gamma_a, \Gamma_b] = i f_{ab}^c \Gamma_c + T_{ab}^i \frac{\delta}{\delta q_i} \ .$$

The second term on the rhs of (2.41) has the form of a *trivial* gauge transformation since $E_i^{(0)} T_{ab}^i \equiv 0$ by construction. If this term is absent, then

$$[\Gamma_a, \Gamma_b] = i f_{ab}^c \Gamma_c \ , \tag{2.42}$$

and one speaks of a "closed algebra". If the structure functions f_{ab}^c depend on the "fields" q_i, one speaks of a "quasi group". In the case where $f_{ab}^c = \text{const.}$, then one has a Lie algebra, and the finite transformations form a Lie group.

Under finite gauge transformations $\vec{q} \to \vec{G}_{(\lambda)}(\vec{q})$, an arbitrary function of $\vec{q}$ transforms as

$$F(\vec{q}) \to F(\vec{G}_{(\lambda)}(\vec{q})) = e^{i\lambda^a \Gamma_a} F(\vec{q}) \ .$$

In particular, for $F = [\frac{\delta S}{\delta q_i}]_{\vec{q} \to \vec{G}_{(\lambda)}(\vec{q})}$ and λ infinitesimal, we have that

$$\left. \frac{\delta S}{\delta q_i} \right|_{q \to G_\epsilon(q)} \approx \left(1 + \epsilon^a R_a^j \frac{\delta}{\delta q_j} \right) \frac{\delta S}{\delta q_i} \ .$$

Making use of the gauge identity (2.40),

$$\left.\frac{\delta S}{\delta q_i}\right|_{\vec{q}\to\vec{G}_\epsilon(\vec{q})} \approx \left(\delta_i^j - \epsilon^a \frac{\delta R_a^j}{\delta q_i}\right)\frac{\delta S}{\delta q_j} \; .$$

Hence it follows that

$$\frac{\delta S}{\delta q_j} = 0 \Longrightarrow \left.\frac{\delta S}{\delta q_j}\right|_{\vec{q}\to\vec{G}_\lambda(\vec{q})} = 0 \; ;$$

i.e., the gauge identity (2.40) implies that if q^i is a possible trajectory, then this will also be the case for $\vec{q} \to \vec{G}_{(\lambda)}(\vec{q})$, for any λ.

Chapter 3

Hamiltonian Approach. The Dirac Formalism

3.1 Introduction

The quantization of non-singular systems is in principle straightforward. The classical Hamilton equations of motion are first written in Poisson bracket form. The corresponding quantum version is then obtained by replacing the Poisson brackets, multiplied by $i\hbar$, by commutators of the corresponding operators. On the other hand, the quantization of singular systems is non-trivial. This chapter is devoted to the introduction of the classical Hamiltonian framework within which such systems can be dealt with. This - as we have already pointed out - is not just an academic exercise, since all gauge field theories fall into the class of singular systems. The following discussion is based on the work of Dirac [Dirac 1950, 1964]. For further literature we refer the reader to [Hansen 1976, Sundermeyer 1982, Gitman 1990, Henneaux/Teitelboim 1992].

3.2 Primary constraints

Consider a system with a finite number of degrees of freedom $\{q_i\}$, whose dynamics is governed by a Lagrangian $L(q,\dot{q}) \equiv L(\{q_i\},\{\dot{q}_i\})$. To arrive at a phase space formulation of the equations of motion one must construct the corresponding Hamiltonian. To this effect one first introduces the momenta canonically conjugate to the "coordinates" q_j,

$$p_j = \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_j} \ . \tag{3.1}$$

For non-singular systems these equations allow one to express the velocities $\dot{q}_i$ in terms of the canonical variables:

$$\dot{q}_i = f_i(q,p)\,.$$

The canonical Hamiltonian is then obtained by performing a Legendre transformation [1]

$$H_c(q,p) = \sum_i p_i f_i(q,p) - L(q, f(q,p)) \ ,$$

where the velocities have been expressed in terms of the canonical momenta and coordinates. The Hamilton equations of motion are then first order in the time derivative and take the standard form.

For the velocities to be expressible in terms of the coordinates and momenta one must have that $det\ W \neq 0$, where the matrix W is the "Hessian" (2.2). In this case, as we have seen in chapter 2, the accelerations $\ddot{q}_i$ are uniquely determined by $\{q_i\}$ and $\{\dot{q}_i\}$. On the other hand, if $det\ W = 0$, then W is not invertible. In this case the Hamilton equations of motion do not take the standard form and one speaks of a singular Lagrangian.

For the time being we shall assume that we are dealing with a finite number of degrees of freedom. The results can then, at least formally, be easily generalized to systems with a non-denumerable number of degrees of freedom.

Consider the Lagrangian

$$L = \frac{1}{2}\sum_{i,j=1}^{n} W_{ij}(q)\dot{q}_i\dot{q}_j + \sum_i \eta_i(q)\dot{q}_i - V(q) \ . \tag{3.2}$$

Since W_{ij} is a symmetric matrix, the canonical momenta are given by

$$p_i = \sum_j W_{ij}(q)\dot{q}_j + \eta_i(q)\,. \tag{3.3}$$

If W is an $n\times n$ singular matrix (i.e. $det\ W = 0$) of rank R_W, then it possesses $n - R_W$ eigenvectors with vanishing eigenvalue. Label these eigenvectors by $\vec{v}^{(\alpha)}$. Then

$$\sum_j W_{ij}(q)v_j^{(\alpha)}(q) = 0 \quad , \quad \alpha = 1,\cdots,n-R_W \ .$$

It then follows from (3.3) that there exist $n - R_W$ constraints

$$\sum_i v_i^{(\alpha)}(q)(p_i - \eta_i(q)) = 0 \ ,$$

[1] In this chapter we use throughout lower indices for convenience, at the expense of introducing explicitly summation signs.

which follow solely from the definition of the canonical momenta (3.1). Let $\{p_a\}$ denote a set of R_W linearly independent momenta. The remaining set of $n-R_W$ momenta we denote with a Greek subscript: $\{p_\alpha\}$. Then the constraint equations will be of the form

$$\sum_\beta M_{\alpha\beta}(q)p_\beta - F_\alpha(q,\{p_a\}) = 0 \tag{3.4}$$

with

$$M_{\alpha\beta}(q) = v_\beta^{(\alpha)}(q)\,,$$

and

$$F_\alpha(q,\{p_b\}) = \sum_i v_i^{(\alpha)}(q)\eta_i(q) - \sum_b v_b^{(\alpha)}(q)p_b\,.$$

M is necessarily an invertible matrix since, if the determinant of M would vanish, then M would possess eigenvectors with vanishing eigenvalues, implying the existence of further constraints. Hence the number of constraints would be larger than $n - R_W$.

Since *det* $M \neq 0$, the constraints (3.4) can also be written in the *standard* form

$$\phi_\alpha(q,p) := p_\alpha - g_\alpha(q,\{p_a\}) = 0\,, \tag{3.5}$$

where

$$g_\alpha = \sum_\beta M^{-1}_{\alpha\beta}F_\beta\ .$$

Written in this form, the label α on the constraint ϕ_α is that of the momentum p_α, conjugate to the velocity $\dot q_\alpha$. The constraints (3.5) are referred to in Dirac's terminology as *primary constraints*. They follow solely from the definition of the canonical momenta, without making use of the equations of motion. The primary constraints have no analog on the Lagrangian level. On the Hamiltonian level they define a $2n-(n-R_W) = n+R_W$-dimensional subspace Γ_P of the 2n-dimensional phase space.

Although the above considerations have been based on a Lagrangian quadratic in the velocities, (3.5) is generally valid for Lagrangians which depend only on q and $\dot q$, and no higher derivatives of q (the Hessian (2.2) can thus in principle also depend on the velocities). This can be seen as follows:

Let W_{ab}, $(a,b = 1,\cdots,R_W)$ be the largest invertible submatrix of W_{ij}, where a suitable rearrangement of the components has been carried out. We can then solve eqs. (3.1) for R_W velocities $\dot q_a$ in terms of the coordinates q_i, the momenta $\{p_a\}$ and the remaining velocites $\{\dot q_\alpha\}$:

$$\dot q_a = f_a(q,\{p_b\},\{\dot q_\beta\})\,,\quad a,b = 1,...,R_W\ ;\ \beta = R_W+1,...,n\,. \tag{3.6}$$

Inserting this expression into (3.1), one arrives at a relation of the form,

$$p_j = h_j(q, \{p_a\}, \{\dot{q}_\alpha\}) \ .$$

For $j = a$ $(a = 1, \cdots, R_W)$ this relation must necessarily reduce to an identity. The remaining equations read

$$p_\alpha = h_\alpha(q, \{p_a\}, \{\dot{q}_\beta\}) \ .$$

But the rhs cannot depend on the velocities $\dot{q}_\beta$, since otherwise we could express still more velocities from the set $\{\dot{q}_\alpha\}$ in terms of the coordinates, the momenta and the remaining velocities, which is not possible.

We now embarque on the construction of the Hamilton equations of motion for such a constraint system, following first a conventional approach, and thereafter another line of reasoning, which emphasizes the role played by the zero modes of W.

3.3 The Hamilton equations of motion

We first prove the following propositions [Sudarshan 1974]:

Proposition 1

On the subspace Γ_P defined by (3.5), the canonical Hamiltonian is only a function of the coordinates $\{q_i\}$ and the momenta $\{p_a\}$; i.e., it does not depend on the velocities $\{\dot{q}_\alpha\}$.

<u>*Proof*</u>

Consider the canonical Hamiltonian on the subspace Γ_P:

$$H_0 \equiv H_c|_{\Gamma_P} = \sum_a p_a f_a + \sum_a g_\alpha \dot{q}_\alpha - L(q, \{f_b\}, \{\dot{q}_\beta\}) \ , \tag{3.7}$$

where we have used (3.5) and (3.6). The rhs of (3.7) defines a function which in principle depends on $\{q_i\}$, $\{p_a\}$ and the velocities $\{\dot{q}_\alpha\}$. We wish to show that H_0 does not depend on the velocities $\{\dot{q}_\alpha\}$. Indeed

$$\begin{aligned}
\frac{\partial H_0}{\partial \dot{q}_\beta} &= \sum_a p_a \frac{\partial f_a}{\partial \dot{q}_\beta} + g_\beta - \left(\frac{\partial L}{\partial \dot{q}_a}\right)_{\dot{q}_a = f_a} \frac{\partial f_a}{\partial \dot{q}_\beta} - \left(\frac{\partial L}{\partial \dot{q}_\beta}\right)_{\dot{q}_a = f_a} \\
&= \sum_a \left(p_a - \left(\frac{\partial L}{\partial \dot{q}_a}\right)_{\dot{q}_a = f_a}\right) \frac{\partial f_a}{\partial \dot{q}_\beta} + \left(g_\beta - \left(\frac{\partial L}{\partial \dot{q}_\beta}\right)_{\dot{q}_a = f_a}\right) = 0 \ .
\end{aligned}$$

Hence it follows that

$$H_0 = H_0(q, \{p_a\})\,. \tag{3.8}$$

Note that we have not used the Lagrange equations of motion in the proof.

Proposition 2

In the presence of primary constraints (3.5), the Hamilton equations of motion read:

$$\begin{aligned}
\dot{q}_i &= \frac{\partial H_0}{\partial p_i} + \sum_\beta \dot{q}_\beta \frac{\partial \phi_\beta}{\partial p_i}\ , \\
\dot{p}_i &= -\frac{\partial H_0}{\partial q_i} - \sum_\beta \dot{q}_\beta \frac{\partial \phi_\beta}{\partial q_i}\ , \\
\phi_\alpha(q,p) &= 0\ ,
\end{aligned} \tag{3.9}$$

where the $\dot{q}_\beta$ are (a priori) undetermined velocities. Note that we are only allowed to set $\phi_\alpha = 0$ after having performed the partial differentiations.

Proof

From (3.7) we obtain, using the fact that H_0 does not depend on $\dot{q}_\alpha$,

$$\frac{\partial H_0}{\partial p_a} = f_a + \sum_b p_b \frac{\partial f_b}{\partial p_a} + \sum_\beta \frac{\partial g_\beta}{\partial p_a}\dot{q}_\beta - \left(\frac{\partial L}{\partial \dot{q}_b}\right)_{\dot{q}_a = f_a} \frac{\partial f_b}{\partial p_a}\ ,$$

or

$$\frac{\partial H_0}{\partial p_a} = \dot{q}_a + \sum_\beta \frac{\partial g_\beta}{\partial p_a}\dot{q}_\beta\,. \tag{3.10}$$

Furthermore

$$\begin{aligned}
\frac{\partial H_0}{\partial q_i} &= \sum_b p_b \frac{\partial f_b}{\partial q_i} + \sum_\beta \frac{\partial g_\beta}{\partial q_i}\dot{q}_\beta - \left(\frac{\partial L}{\partial q_i}\right)_{\dot{q}_a = f_a} - \sum_b \left(\frac{\partial L}{\partial \dot{q}_b}\right)_{\dot{q}_b = f_b} \frac{\partial f_b}{\partial q_i} \\
&= -\left(\frac{\partial L}{\partial q_i}\right)_{\{\dot{q}_a = f_a\}} + \sum_\beta \frac{\partial g_\beta}{\partial q_i}\dot{q}_\beta \\
&= -\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)_{\{\dot{q}_a = f_a\}} + \sum_\beta \frac{\partial g_\beta}{\partial q_i}\dot{q}_\beta\,,
\end{aligned}$$

or

$$\frac{\partial H_0}{\partial q_i} = -\dot{p}_i + \sum_i \frac{\partial g_\beta}{\partial q_i}\dot{q}_\beta\,. \tag{3.11}$$

Note that this time we have made use of the Euler-Lagrange equations of motion.

Equations (3.10) and (3.11) are asymmetric in the number of coordinates and momenta, and their time derivatives. A more symmetric form is obtained by noting that H_0 does not depend on $\{p_\alpha\}$. Making use of the definition of ϕ_α (cf. eq. (3.5)) we first rewrite the above equations as follows:

$$\dot{q}_a = \frac{\partial H_0}{\partial p_a} + \sum_\beta \dot{q}_\beta \frac{\partial \phi_\beta}{\partial p_a} \ , \tag{3.12}$$

$$\dot{p}_i = -\frac{\partial H_0}{\partial q_i} - \sum_\beta \dot{q}_\beta \frac{\partial \phi_\beta}{\partial q_i} \ . \tag{3.13}$$

Because $\frac{\partial H_0}{\partial p_\alpha} = 0$, and $\frac{\partial \phi_\beta}{\partial p_\alpha} = \delta_{\beta\alpha}$, we can supplement these equations by the identities

$$\dot{q}_\alpha \equiv \frac{\partial H_0}{\partial p_\alpha} + \sum_\beta \dot{q}_\beta \frac{\partial \phi_\beta}{\partial p_\alpha} \ .$$

Adjoining these equations to (3.12), we arrive at (3.9).

It is not obvious that the equations for $\dot{q}_i$ and $\dot{p}_i$ are consistent with the primary constraints (3.5). Thus, for consistency we must also have

$$\dot{p}_\alpha = \frac{d}{dt} g_\alpha(q, \{p_a\}) \ ,$$

where $\dot{p}_\alpha$ is given by the rhs of (3.13), together with the primary constraints. This may lead to so-called "secondary constraints". These constraints must in turn be consistent with the equations of motion and primary constraints (and so forth). We shall soon look at this problem in more detail. In fact, as we will see, the full set of Lagrange equations of motion will only become manifest on Hamiltonian level once secondary constraints are taken into account. This does not mean, however, that they must be added explicitly to the Hamilton equations of motion (3.9), since they are hidden in the primary constraints, which are required to hold for all times.

3.3.1 Streamlining the Hamilton equations of motion

For the following discussion it is useful to introduce the concept of a weak equality as defined by Dirac:

Definition: f is weakly equal to g ($f \approx g$) if f and g are equal on the subspace defined by the primary constraints $\phi_\beta = 0$: $f|_{\Gamma_P} = g|_{\Gamma_P}$.

In the following $f(q,p) \approx g(q,p)$ will thus be a short hand for the following set of equations,

$$f(q,p) = g(q,p) \ , \quad \phi_\alpha(q,p) = 0 \ .$$

With this definition we can write the Hamilton equations (3.9) in the form

$$\dot{q}_i \approx \frac{\partial H_T}{\partial p_i} \, , \quad \dot{p}_i \approx -\frac{\partial H_T}{\partial q_i} \, , \tag{3.14}$$

where the "total Hamiltonian" H_T is defined by

$$H_T = H_0 + \sum_\alpha v_\alpha \phi_\alpha \ . \tag{3.15}$$

Note that the derivatives in (3.14) do not act on the v_α since the expressions are to be evaluated on the primary constrained surface, as is implied by the "weak equality" sign. In fact, since the "velocities" v_α are a priori not given functions of the canonical variables, the action of the derivatives would not even be defined.

Making use of the definition of "weak equality", we can further relax the definition of H_T. To this end we first prove the following proposition: [Sudarshan 1974]:

Proposition 3

If f and h are two functions defined over the entire phase space Γ, spanned by $\{q_i\}$ and $\{p_i\}$, and if $f(q,p)|_{\Gamma_P} = h(q,p)|_{\Gamma_P}$, then

$$\begin{aligned}
\frac{\partial}{\partial q_i}\left(f - \sum_\beta \phi_\beta \frac{\partial f}{\partial p_\beta}\right) &\approx \frac{\partial}{\partial q_i}\left(h - \sum_\beta \phi_\beta \frac{\partial h}{\partial p_\beta}\right) \\
\frac{\partial}{\partial p_i}\left(f - \sum_p \phi_\beta \frac{\partial f}{\partial p_\beta}\right) &\approx \frac{\partial}{\partial p_i}\left(h - \sum_\beta \phi_\beta \frac{\partial h}{\partial p_\beta}\right)
\end{aligned} \tag{3.16}$$

<u>*Proof:*</u>

Consider two functions $f(q,\{p_a\},\{p_\alpha\})$ and $h(q,\{p_a\},\{p_\alpha\})$. By assumption,

$$f(q,\{p_a\},\{g_\alpha\}) = h(q,\{p_a\},\{g_\alpha\})$$

where we made use of (3.5). From here it follows that

$$\left(\frac{\partial f}{\partial q_i} + \sum_\beta \frac{\partial f}{\partial p_\beta}\frac{\partial g_\beta}{\partial q_i}\right)_{\Gamma_P} = \left(\frac{\partial h}{\partial q_i} + \sum_\beta \frac{\partial h}{\partial p_\beta}\frac{\partial g_\beta}{\partial q_i}\right)_{\Gamma_P}$$

and

$$\left(\frac{\partial f}{\partial p_a} + \sum_\beta \frac{\partial f}{\partial p_\beta}\frac{\partial g_\beta}{\partial p_a}\right)_{\Gamma_P} = \left(\frac{\partial h}{\partial p_a} + \sum_\beta \frac{\partial h}{\partial p_\beta}\frac{\partial g_\beta}{\partial p_a}\right)_{\Gamma_P} .$$

Making use of (3.5), and of the definition of the weak equality, these equations can also be written in the form

$$\frac{\partial}{\partial q_i}\left(f - \sum_\beta \phi_\beta \frac{\partial f}{\partial p_\beta}\right) \approx \frac{\partial}{\partial q_i}\left(h - \sum_\beta \phi_\beta \frac{\partial h}{\partial p_\beta}\right) ,$$

$$\frac{\partial}{\partial p_a}\left(f - \sum_\beta \phi_\beta \frac{\partial f}{\partial p_\beta}\right) \approx \frac{\partial}{\partial p_a}\left(h - \sum_\beta \phi_\beta \frac{\partial h}{\partial p_\beta}\right) .$$

Finally, since $\frac{\partial \phi_\beta}{\partial p_\alpha} = \delta_{\beta\alpha}$ we also trivially have that

$$\frac{\partial}{\partial p_\alpha}\left(f - \sum_\beta \phi_\beta \frac{\partial f}{\partial p_\beta}\right) \approx \frac{\partial}{\partial p_\alpha}\left(h - \sum_\beta \phi_\beta \frac{\partial h}{\partial p_\beta}\right) .$$

Hence we are led to (3.16).

<u>Corollary:</u>

From (3.16) it follows in particular that

$$\begin{aligned} \dot{q}_i &\approx \frac{\partial H}{\partial p_i} + \sum_\beta v^\beta \frac{\partial \phi_\beta}{\partial p_i} , \\ \dot{p}_i &\approx -\frac{\partial H}{\partial q_i} - \sum_\beta v_\beta \frac{\partial \phi_\beta}{\partial q_i} , \end{aligned} \tag{3.17}$$

where $H \approx H_0$, and v^β are undetermined parameters.

<u>Proof:</u>

Consider two functions $H(\{q_i\},\{p_i\})$ and $H_0(\{q_i\},\{p_a\})$, satisfying

$$H(\{q_i\},\{p_i\}) \approx H_0(\{q_i\},\{p_a\}).$$

Setting $f = H_0$ and $h = H$ in (3.16), it follows from (3.8) that

$$\frac{\partial H_0}{\partial q_i} \approx \frac{\partial}{\partial q_i}\left(H - \sum_\rho \phi_\beta \frac{\partial H}{\partial p_\beta}\right) ,$$

$$\frac{\partial H_0}{\partial p_i} \approx \frac{\partial}{\partial p_i}\left(H - \sum_\rho \phi_\beta \frac{\partial H}{\partial p_\beta}\right) .$$

Inserting these expressions in (3.9), we obtain

$$\dot{q}_i \approx \frac{\partial}{\partial p_i}\left(H - \sum_\beta \phi_\beta \frac{\partial H}{\partial p_\beta}\right) + \sum_\beta \dot{q}_\beta \frac{\partial \phi_\beta}{\partial p_i} \ ,$$

$$\dot{p}_i \approx -\frac{\partial}{\partial q_i}\left(H - \sum_\beta \phi_\beta \frac{\partial H}{\partial p_\beta}\right) - \sum_\beta \dot{q}_\beta \frac{\partial \phi_\beta}{\partial q_i} \ .$$

These expressions can also be written in the form [2]

$$\dot{q}_i \approx \frac{\partial}{\partial p_i}\left[H + \sum_\beta \left(\dot{q}_\beta - \frac{\partial H}{\partial p_\beta}\right)\phi_\beta\right] \ ,$$

$$\dot{p}_i \approx -\frac{\partial}{\partial q_i}\left[H + \sum_\beta \left(\dot{q}_\beta - \frac{\partial H}{\partial p_\beta}\right)\phi_\beta\right] \ .$$

Defining

$$v_\beta := \dot{q}_\beta - \frac{\partial H}{\partial p_\beta} \ , \tag{3.18}$$

equations (3.17) follow. These equations can also be written in the form

$$\dot{q}_i \approx \frac{\partial H_T}{\partial p_i} \ , \quad \dot{p}_i \approx -\frac{\partial H_T}{\partial q_i} \ , \tag{3.19}$$

where H_T is the *total* Hamiltonian (3.15) with H_0 replaced by H:

$$H_T = H + \sum_\beta v_\beta \phi_\beta \ \ , \ H \approx H_0 \ . \tag{3.20}$$

Hence the Hamilton equations of motion take a form analogous to those for an unconstrained system. The canonical Hamiltonian is merely replaced by the total Hamiltonian (3.20), where $H(q,p)$ is now any function of the canonical variables which on the primary constrained surface Γ_P reduces to the canonical Hamiltonian evaluated on this surface. Note that in the case where $H = H_0$ we have from (3.18) and (3.8) that $v_\beta = \dot{q}_\beta$.

Equations (3.19) can also be written in Poisson bracket form. Defining the Poisson bracket of $A(q,p)$ and $B(q,p)$ in the usual way, [3]

$$\{A, B\} = \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right) \tag{3.21}$$

[2] Note again that the action of the partial derivatives on the $\dot{q}_\alpha$'s is not defined. Such terms are however proportional to the constraints and therefore vanish on the primary surface.

[3] In this chapter we restrict ourselves to commuting c-number variables.

we have that

$$\dot{q}_i \approx \{q_i, H_T\} \ , \quad \dot{p}_i \approx \{p_i, H_T\} \ . \tag{3.22}$$

For later purposes we recall some fundamental properties of Poisson brackets:

i) *Antisymmetry*

$$\{f, g\} = -\{g, f\}$$

ii) *Linearity*

$$\{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\}$$

iii) *Associative product law*

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + \{f_1, g\}f_2$$

iv) *Jacobi-identity (bosonic)*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \ . \tag{3.23}$$

From these properties it follows that for any function of the phase space variables $F(q, p)$,

$$\dot{F} \approx \{F, H_T\} \ .$$

3.3.2 Alternative derivation of the Hamilton equations

Before proceeding with our Hamiltonian analysis of singular systems, it is instructive to rederive the above equations of motion by starting from a non-singular Lagrangian depending on one or more parameters, for which the Hamilton equations of motion are the standard ones, and which reduces to the singular Lagrangian of interest by taking an appropriate limit [Rothe 2003a]. This, as we shall see, will also shed light on the role played by the primary constraints.

Consider the following Lagrangian,

$$L = \frac{1}{2}\sum_{i,j} W_{ij}(q; \alpha)\dot{q}_i\dot{q}_j - \sum_i \eta_i(q)\dot{q}_i - V(q) \ , \tag{3.24}$$

where q stands for the set of coordinates $\{q_i\}$, and α stands collectively for a set of parameters. For $\alpha \neq \alpha_c$ we assume that *det* $W \neq 0$, so that we are dealing with a non-singular system. We assume that the singular system of interest is realized for $\alpha = \alpha_c$ (the subscript c stands for "critical"), where *det* $W(q; \alpha_c) = 0$. [4] Our aim is to derive the Hamilton equations of motion

[4] Clearly the form of (3.24) is not unique.

for the singular system, by taking the limit $\alpha \to \alpha_c$ of the Hamilton equations following from (3.24).

For $\alpha \neq \alpha_c$, one has for the Legendre transform of (3.24) the canonical Hamiltonian,

$$H_c = \frac{1}{2} \sum_{i,j} (p_i + \eta_i) W^{-1}_{ij} (p_j + \eta_j) + V(q) \ , \tag{3.25}$$

where the canonical momenta are related to the velocities by

$$p_i = \sum_j W_{ij}(q;\alpha)\dot{q}_j - \eta_i(q) \ . \tag{3.26}$$

The symmetric matrix W can be diagonalized by an orthogonal transformation $W \to W_D = C^T W C$. In terms of the eigenvalues and orthonormalized eigenvectors, W can be written in the form

$$W_{ij} = \sum_\ell \lambda_\ell \, v_i^{(\ell)} v_j^{(\ell)} \ ,$$

where

$$W(q;\alpha)\vec{v}^{(\ell)}(q;\alpha) = \lambda_\ell(q;\alpha)\vec{v}^{(\ell)}(q;\alpha) \ , \tag{3.27}$$

and where by assumption, $\lambda_\ell(q;\alpha) \neq 0$ for $\alpha \neq \alpha_c$. Correspondingly W^{-1}_{ij} is obtained by making the replacement $\lambda_\ell \to \frac{1}{\lambda_\ell}$. The Hamiltonian (3.25) is thus given by

$$H_c = \frac{1}{2} \sum_\ell \frac{1}{\lambda_\ell} \phi_\ell^2 + V(q) \ , \tag{3.28}$$

where

$$\phi_\ell(q,p;\alpha) := (\vec{p} + \vec{\eta}(q)) \cdot \vec{v}^{(\ell)}(q;\alpha) \ . \tag{3.29}$$

For $\alpha \neq \alpha_c$ the Hamilton equations of motion then take the form

$$\dot{q}_i = \frac{\partial H_c}{\partial p_i} = \sum_\ell \frac{1}{\lambda_\ell} \phi_\ell \frac{\partial \phi_\ell}{\partial p_i} \ , \tag{3.30}$$

$$\dot{p}_i = -\frac{\partial H_c}{\partial q_i} = -\frac{1}{2} \sum_\ell \frac{\partial}{\partial q_i} \left(\frac{1}{\lambda_\ell} \phi_\ell^2 \right) - \frac{\partial V}{\partial q_i} \ . \tag{3.31}$$

Consider now the limit $\alpha \to \alpha_c$ where *det* $W \to 0$. Let $\{\lambda_{\ell_0}\}$ denote the set of eigenvalues which vanish in this limit. In order to implement the limit we first write (3.30) and (3.31) in the form

$$\dot{q}_i = \sum_{\ell \neq \{\ell_0\}} \frac{1}{\lambda_\ell} \phi_\ell \frac{\partial \phi_\ell}{\partial p_i} + \sum_{\ell_0} \frac{1}{\lambda_{\ell_0}} \phi_{\ell_0} \frac{\partial \phi_{\ell_0}}{\partial p_i} \ , \tag{3.32}$$

$$\dot{p}_i = -\frac{1}{2}\sum_{\ell\neq\{\ell_0\}}\frac{\partial}{\partial q_i}\left(\frac{1}{\lambda_\ell}\phi_\ell^2\right) - \frac{1}{2}\sum_{\ell_0}\frac{\partial}{\partial q_i}\left(\frac{1}{\lambda_{\ell_0}}\phi_{\ell_0}^2\right) - \frac{\partial V}{\partial q_i}\ . \tag{3.33}$$

From (3.29), (3.26) and (3.27) we have that

$$\phi_{\ell_0} = \sum_{i,j} v_i^{(\ell_0)} W_{ij}\dot{q}_j = \lambda_{\ell_0}(\vec{v}^{(\ell_0)}\cdot\dot{\vec{q}})\ . \tag{3.34}$$

Hence

$$\lim_{\alpha\to\alpha_c}\frac{1}{\lambda_{\ell_0}}\phi_{\ell_0} = \rho_{\ell_0}(q,\dot{q};\alpha_c)\ , \tag{3.35}$$

where

$$\rho_{\ell_0}(q,\dot{q};\alpha_c) = \dot{\vec{q}}\cdot\vec{v}^{(\ell_0)}(q;\alpha_c)\ . \tag{3.36}$$

Note that the finiteness of the velocities in (3.34) implies that

$$\phi_{\ell_0}(q,p,\alpha_c) = 0\ , \tag{3.37}$$

where ϕ_{ℓ_0} is given by (3.29) with $\ell = \ell_0$. These are just the primary constraints.

Hence for $\alpha\to\alpha_c$, equation (3.32) reduces to

$$\dot{q}_i = \sum_{\ell\neq\{\ell_0\}}\left(\frac{1}{\lambda_\ell}\phi_\ell\frac{\partial\phi_\ell}{\partial p_i}\right)_{\alpha=\alpha_c} + \sum_{\ell_0}\rho_{\ell_0}\left(\frac{\partial\phi_{\ell_0}}{\partial p_i}\right)_{\alpha=\alpha_c} \tag{3.38}$$

with ρ_{ℓ_0} given by (3.36). Consider next eqs. (3.33). Again making use of (3.35) and the fact that $\frac{\partial\lambda_{\ell_0}}{\partial q_i}|_{\alpha=\alpha_c} = 0$, these reduce for $\alpha\to\alpha_c$ to

$$\dot{p}_i = -\frac{1}{2}\sum_{\ell\neq\{\ell_0\}}\left[\frac{\partial}{\partial q_i}\left(\frac{1}{\lambda_\ell}\phi_\ell^2\right)\right]_{\alpha=\alpha_c} - \sum_{\ell_0}\rho_{\ell_0}\left(\frac{\partial\phi_{\ell_0}}{\partial q_i}\right)_{\alpha=\alpha_c} - \frac{\partial V}{\partial q_i}\ , \tag{3.39}$$

where use has been made of (3.36).

We now notice that the first sum on the rhs of eqs. (3.38) and (3.39) are just given by $\frac{\partial H_0}{\partial p_i}$ and $-\frac{\partial H_0}{\partial q_i}$, where H_0 is the Hamiltonian obtained from (3.28) by taking the limit $\alpha\to\alpha_c$:

$$H_0 = \frac{1}{2}\sum_{\ell\neq\ell_0}\frac{1}{\lambda_\ell(q,\alpha_c)}\phi_\ell^2(q,p;\alpha_c) + V(q) \tag{3.40}$$

Here use has again been made of the fact that $\lim_{\alpha\to\alpha_c}\frac{\phi_{\ell_0}}{\lambda_{\ell_0}}$ is finite, as well as of (3.37). H_0 is just the canonical Hamiltonian derived from the (3.24) with $\alpha = \alpha_c$, evaluated on the surface (3.37). Hence, together with the constraint equations (3.37), the equations of motion (3.38) and (3.39) take the form

$$\dot{q}_i \approx \frac{\partial H_T}{\partial p_i}\ , \quad \dot{p}_i \approx -\frac{\partial H_T}{\partial q_i} \tag{3.41}$$

where

$$H_T = H_0 + \sum_{\ell_0} \rho_{\ell_0}\phi_{\ell_0} \ ,$$

is the total Hamiltonian, and ρ_{ℓ_0} are the projections of the velocities on the zero modes (3.36). Hence we have obtained the Hamilton equations of motion for a constrained system in the form given by Dirac, by taking the limit $\alpha \to \alpha_c$ of the equations of motion for an unconstrained system. The primary constraints are just the statement that the projected velocities (3.36) are finite in this limit.

The primary constraints themselves have no analog on the Lagrangian level but allow us to recover the connection between momenta and velocities (3.26) for $\alpha = \alpha_c$, needed to express the Hamilton equations of motion in terms of Lagrangian variables. Thus from (3.38), (3.36) and (3.29) it follows that

$$\dot{q}_i = \sum_{\ell \neq \{\ell_0\}} \frac{1}{\lambda_\ell}\phi_\ell v_i^{(\ell)} + \sum_{\ell_0} (\dot{\vec{q}} \cdot v^{(\ell_0)}) v_i^{(\ell_0)} \ , \tag{3.42}$$

where the rhs is evaluated at $\alpha = \alpha_c$. Define the matrix $W'_{ij} = \sum_{\ell \neq \{\ell_0\}} \lambda_\ell v_i^{(\ell)} v_j^{(\ell)}$. From (3.42), and making use of (3.29) and (3.37) we have,

$$\begin{aligned} \sum_j W'_{ij}(q,\alpha_c)\dot{q}_j &= \sum_{\ell \neq \ell_0} \phi_\ell(q,p,\alpha_c) v_i^{(\ell)} \\ &= \sum_{\ell,j} (p+\eta)_j v_i^{(\ell)} v_j^{(\ell)} = (p+\eta)_i \ , \end{aligned}$$

where in the last step we made use of the primary constraints, in order to extend the sum on the rhs over all ℓ, and have then used the completeness relation for the eigenvectors. Since $W' = W(q,\alpha_c)$ we have thus recovered (3.26).

Equations (3.41), together with the primary constraints, do not directly yield the complete set of Lagrange equations of motion. These follow by also requiring the persistance in time of the primary constraints, which may lead to secondary constraints that must in turn be satisfied for all times. The persistence of the primary constraints yields on Lagrange level equations involving only coordinates and velocities. These are part of the Euler-Lagrange equations of motion. From the point of view taken in this section, that the equations of motion are obtained by a limiting procedure from those of an unconstrained system, the persistence of the primary constraints follow from the requirement that also the accelerations remain finite in the limit $\alpha \to \alpha_c$. Thus for $\alpha \neq \alpha_c$ we have from (3.34) that $\phi_{\ell_0} = \lambda_{\ell_0}\dot{\vec{q}} \cdot \vec{v}^{(\ell_0)}$. Taking the time derivative of this expression, and noting that $\lambda_{\ell_0}(q,\alpha)$, and $\partial_i\lambda_{\ell_0}(q,\alpha)$ both vanish in the limit $\alpha \to \alpha_c$, we conclude that $\dot{\phi}_{\ell_0}(q,p,\alpha_c) = 0$. This requirement must be imposed to yield the missing Euler-Lagrange equation.

3.3.3 Examples

Let us consider some simple, but instructive examples.

Example 1

Consider the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x-y)^2 \,. \tag{3.43}$$

This Lagrangian is singular since W has rank $R_W = 1$. The Lagrange equations of motion read

$$\ddot{x} = y - x - \dot{y} \ , \tag{3.44}$$

$$\dot{x} = y - x \ . \tag{3.45}$$

The last equation relates the velocity $\dot{x}$ to the coordinates at any time. It therefore has the form of a constraint. In fact, as we shall see below it corresponds to a so-called secondary constraint on Hamiltonian level.

So far only $\dot{x}$ is determined in terms of the coordinates. But the consistency of the above two equations leads to

$$\dot{y} = y - x \ . \tag{3.46}$$

Note that this equation is not part of the Euler-Lagrange equations of motion, but is implied by their consistency. We will see soon what is the analogous statement on the Hamiltonian level. Although both equations, (3.45) and (3.46) relate velocities to the coordinates at arbitrary time, and therefore appear to be constraints, only the first one is actually a constraint on Hamiltonian level, as we shall see below.

To arrive at a Hamiltonian formulation we first compute the canonical momenta,

$$p_x = \dot{x} + y \ , \quad p_y = 0 \ .$$

The equation $p_y = 0$ is a primary constraint, since it follows directly from the definition of the canonical momenta. The canonical Hamiltonian evaluated on the constrained surface $p_y = 0$ reads

$$H_0 = \frac{1}{2}(p_x - y)^2 + \frac{1}{2}(x-y)^2 \ ,$$

and the total Hamiltonian H_T is given by

$$H_T = H_0 + v p_y \ .$$

From (3.14) we are thus led to the following equations of motion

$$\dot{y} = v \tag{3.47}$$

$$\dot{x} = p_x - y\,, \quad \dot{p}_x = y - x \tag{3.48}$$

$$\dot{p}_y = p_x + x - 2y \tag{3.49}$$

$$p_y = 0\,. \tag{3.50}$$

Equations (3.48) are equivalent to the Lagrange equation of motion (3.44). The other Euler Lagrange equation (3.45) does not appear at this stage. But by requiring that the primary constraint (which has no analog in the Lagrangian formulation) must hold for all times, i.e. $\dot{p}_y \approx \{p_y, H_T\} \approx 0$, and making use of (3.49), we generate a new constraint

$$p_x + x - 2y = 0\,. \tag{3.51}$$

According to Dirac's terminology this is a *secondary* constraint. Equation (3.48) now allows us to rewrite the above equation in terms of Lagrangian variables. This leads to (3.45), so that we retrieved the complete set of Lagrangian equations of motion. On the other hand, eq. (3.46) is not one of the Lagrange equations of motion, but is merely implied by their consistency. On the Hamiltonian level this equation follows by requiring the persistence in time of the secondary constraint, i.e., $\dot{p}_x + \dot{x} - 2\dot{y} = 0$. Together with (3.48) and (3.45), this leads to (3.46), which in turn fixes that Lagrange multiplier v in (3.47). Thus in this example the persistence in time of the secondary constraint did not produce a new constraint, but rather fixed the Lagrange multiplier, which now becomes a function of the canonical variables. [5]

Example 2

Let us now see what happens if we merely change the sign of the last term on the rhs of the Lagrangian (3.43):

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x-y)^2\,. \tag{3.52}$$

This is again a singular Lagrangian involving one primary constraint $p_y = 0$. The total Hamiltonian is given by

$$H_T = \frac{1}{2}p_x^2 - \frac{1}{2}x^2 - yp_x + xy + vp_y\,.$$

[5] As we shall see soon, this is due to the fact that the constraints generated form a so-called "second class" system.

The equations of motion, together with the primary constraint, read

$$\begin{aligned}
&\dot{x} = p_x - y \ , \quad \dot{y} = v \ , \\
&\dot{p}_x = x - y \ , \quad \dot{p}_y = p_x - x \\
&p_y = 0 \ .
\end{aligned}$$

Persistence of the primary constraint leads to the secondary constraint

$$p_x - x = 0 \ .$$

Requiring persistence of this secondary constraint just reproduces the same constraint, so that there are no further constraints. Note that in contrast to the previous example, the velocity $\dot{y}$ is left undetermined. The origin of this was a simple reversal in the sign of the last term in (3.43). The fact that the velocity $\dot{y}$ is left undetermined signalizes, as we shall see later, that the Lagrange equations of motion possess a local symmetry.

Example 3

Consider the Lagrangian (2.6), i.e.,

$$L = \frac{1}{2}(\dot{q}_2 - e^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2 \ .$$

The Lagrange equations of motion read

$$\begin{aligned}
&\dot{q}_2 = e^{q_1} \ , \quad \ddot{q}_3 - \dot{q}_2 = 0 \ , \\
&\frac{d}{dt}(\dot{q}_2 - e^{q_1}) = q_2 - \dot{q}_3 \ .
\end{aligned}$$

They are equivalent to the set

$$\begin{aligned}
&\dot{q}_2 - e^{q_1} = 0 \ , \\
&\dot{q}_3 - q_2 = 0 \ .
\end{aligned} \tag{3.53}$$

The canonical momenta are given by

$$\begin{aligned}
&p_1 = 0 \ , \\
&p_2 = \dot{q}_2 - e^{q_1} \ , \\
&p_3 = \dot{q}_3 - q_2 \ .
\end{aligned} \tag{3.54}$$

The first equation represents a primary constraint. The total Hamiltonian is thus given by

$$H_T = \frac{1}{2}p_2^2 + \frac{1}{2}p_3^2 + e^{q_1}p_2 + q_2 p_3 + v p_1 \ .$$

The Hamilton equations of motion read

$$\begin{aligned} &\dot{q}_1 = v \quad ; \quad \dot{q}_2 = p_2 + e^{q_1} \quad ; \quad \dot{q}_3 = p_3 + q_2 \ , \\ &\dot{p}_1 = -e^{q_1} p_2 \quad ; \quad \dot{p}_2 = -p_3 \quad ; \quad \dot{p}_3 = 0 \ , \end{aligned} \tag{3.55}$$

together with the primary constraint $p_1 = 0$. From the persistence of the primary constraint, i.e. $\dot{p}_1 = 0$, we conclude from (3.55) that $p_2 = 0$. This is a secondary constraint. Note that this new constraint generates via (3.54) the first Lagrange equation of motion in (3.53), as expected. Persistence of the secondary constraint leads, according to (3.55) to the tertiary constraint $p_3 = 0$. Again making use of (3.54) this yields the second Lagrange equation of motion in (3.53). Hence in this example, the primary, secondary and tertiary constraints yield the full set of Lagrange equations of motion, and the Lagrange multiplier v in the Hamilton equations of motion remains undetermined. [6] We emphasize that, although the secondary constraints are hidden in the Hamilton equations of motion, which include in principle only the primary constraints, it will be important to make all the constraints manifest before embarking on the quantization of the theory.

So far we have considered systems with only a few degrees of freedom. The following example illustrates the case of a system with a non-denumerable number of degrees of freedom. At the same time it demonstrates how the Hamilton equations of motion can be derived by a limiting procedure from a non-singular Lagrangian, as discussed in the previous section.

Example 4

Consider the singular Lagrangian of the pure $U(1)$ Maxwell theory in the absence of sources:

$$L[A^\mu, \dot{A}^\mu] = -\frac{1}{4} \int d^3x F_{\mu\nu} F^{\mu\nu} \ , \tag{3.56}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The discrete index i in q_i is now replaced by a continuous index $\vec{x}$ and a discrete Lorentz index, i.e. $q_i(t) \to A_\mu(\vec{x}, t)$.

Consider further the non-singular Lagrangian

$$L[A^\mu, \dot{A}^\mu] = -\frac{1}{4} \int d^3x F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}\alpha \int d^3x \ (\dot{A}^0)^2 \ ,$$

which for $\alpha \to 0$ reduces to (3.56). This (non-covariant) choice is of course only a question of simplicity. Any other non-singular Lagrangian reducing to (3.56)

[6] All constraints which are not primary, will be referred to from here on simply as secondary constraints.

in the appropriate limit would be just as acceptable. The canonical momenta conjugate to A^μ are given by

$$\pi_\mu = F^{0\mu} + \alpha\delta_{\mu 0}\partial_0 A^0 \ .$$

The Lagrangian written in the analogous form to (3.24) is

$$L = \int d^3x \; d^3y \; \left[\frac{1}{2}\dot{A}^\mu(x)W_{\mu\nu}(x,y)\dot{A}^\nu(y)\right] - \int d^3x \; \eta_\mu(A(x))\dot{A}^\mu(x) - V[A]$$

where $x^0 = y^0$,

$$\eta_\mu := \left(0, -\vec{\nabla}A^0\right) \ ,$$

and the matrix elements of the symmetric matrix W read

$$\begin{aligned} W_{0i}(\vec{x},\vec{y}) &= 0 \ , \\ W_{00}(\vec{x},\vec{y}) &= \alpha\delta^{(3)}(\vec{x}-\vec{y}) \ , \\ W_{ij}(\vec{x},\vec{y}) &= \delta_{ij}\delta^{(3)}(\vec{x}-\vec{y}) \ . \end{aligned}$$

The potential $V[A]$ is given by

$$V[A] = \int d^3x \left(\frac{1}{4}F_{ij}F^{ij} - \frac{1}{2}(\vec{\nabla}A^0)^2\right) \ ,$$

and the canonical momenta, analogous to (3.26) are

$$\pi_\mu(x) = \int d^3y \; W_{\mu\nu}(x,y)\dot{A}^\nu(y) - \eta_\mu(A(x)) \ .$$

The matrix W possesses the following eigenvectors, $\vec{v}^{(\lambda,\vec{z})}$, labeled by a discrete and continuous index,

$$v_\nu^{(\lambda,\vec{z})}(\vec{x}) = \delta_{\lambda\nu}\delta^{(3)}(\vec{x}-\vec{z}) \ , \tag{3.57}$$

and the corresponding eigenvalues of W are given by

$$\lambda^{(0,\vec{z})} = \alpha\, ; \quad \lambda^{(i,\vec{z})} = 1 \ , \quad (i = 1,2,3) \ .$$

The canonical Hamiltonian takes the form

$$\begin{aligned} H_c = &\frac{1}{2}\sum_i \int d^3z \left[\int d^3x \, (\pi_\mu(x) + \eta_\mu(x)) \, v_\mu^{(i,\vec{z})}(\vec{x})\right]^2 \\ &+ \frac{1}{2}\int d^3z \frac{1}{\alpha}\left[\int d^3x \, (\pi_\mu(x) + \eta_\mu(x)) \, v_\mu^{(0,\vec{z})}(\vec{x})\right]^2 + V[A] \ , \end{aligned}$$

where $\eta_\mu(x) \equiv \eta_\mu(A(x))$. This is the analog of (3.28). Upon making use of (3.57), H_c reduces to

$$H_c = \frac{1}{2}\sum_i \int d^3x \; (\pi_i + \eta_i)^2 + \frac{1}{2}\int d^3x \; \frac{1}{\alpha}(\pi_0 + \eta_0)^2 + V[A] \; ,$$

and the Hamilton equations of motion read

$$\begin{aligned}
\dot{A}^0 &= \frac{\delta H_c}{\delta \pi_0(x)} = \frac{\pi_0}{\alpha} \; , \\
\dot{A}^i &= \frac{\delta H_c}{\delta \pi_i(x)} = \pi_i + \partial^i A^0 \; , \\
\dot{\pi}_0 &= -\frac{\delta H_c}{\delta A^0(x)} = -\vec{\nabla}\cdot\vec{\pi} \; , \\
\dot{\pi}_i &= -\frac{\delta H_c}{\delta A^i(x)} = -\partial_j F^{ji} \; .
\end{aligned}$$

For finite $\dot{A}^0$ the first equation tells us that π_0 vanishes in the limit $\alpha \to 0$ ($\pi_0 = 0$ is a primary constraint), whereas $\dot{A}^0$ remains completely arbitrary. Since π_0 must vanish for arbitrary times, the third equation tells us that also $\vec{\nabla}\cdot\vec{\pi} = 0$. This is just the secondary constraint (Gauss's law).

Alternatively we could have departed from a covariant form for an unconstrained system by adding to (3.56) a term $\frac{1}{2}\alpha\int d^3x \; (\partial_\mu A^\mu)^2$. In this case $\eta_\mu(A(x))$ would also depend on α. Following our general procedure one is then led to the equations of motion

$$\begin{aligned}
\dot{A}^0 &= \frac{\partial H_c}{\partial \pi_0} = \frac{1}{\alpha}\pi_0 - \vec{\nabla}\cdot\vec{A} \; , \\
\dot{A}^i &= \frac{\partial H_c}{\partial \pi_i} = \pi_i + \partial^i A^0 \; , \\
\dot{\pi}_0 &= -\frac{\partial H_c}{\partial A^0} = -\vec{\nabla}\cdot\vec{\pi} \; , \\
\dot{\pi}_i &= -\partial_j F^{ji} + \partial^i \pi_0 \; .
\end{aligned}$$

From the first equation it follows again that in the limit $\alpha \to 0$, $\pi_0 = 0$. This is the primary constraint. The remaining equations then reduce to the standard equations of the Maxwell theory. In particular, since $\dot{\pi}_0 = 0$, we have that $\vec{\nabla}\cdot\vec{\pi} = 0$. This is Gauss's law in the absence of sources, and is a secondary constraint following from the time persistence of the primary constraint.

3.4 Iterative procedure for generating the constraints

As we have emphasized above, the complete dynamics of a system is described by the Hamilton equations of motion (3.19). The above examples have shown that further (secondary) constraints may be hidden in these equations, and that they are part of the Lagrange equations of motion, or follow from their consistency. As we have also mentioned, it is important to unravel all the constraints in order to formulate the corresponding quantum theory. In this section we discuss in more detail the Dirac algorithm for generating the secondary constraints in a systematic way [Dirac 1964]. The starting point is always given by the primary constraints

$$\phi_\beta = 0, \quad \beta = 1, ..., n - R_W$$

which are required to hold for all times, i.e.

$$\dot{\phi}_\beta = 0.$$

Recall that n is the number of coordinate degrees of freedom, and that R_W is the rank of the matrix W defined in (2.2). The persistence in time of the constraints can only be required in the weak sense, i.e., one should implement any constraint only after having carried out the partial differentiations in (3.19), or the Poisson brackets in (3.22):

$$\dot{\phi}_\beta \approx \{\phi_\beta, H_T\} \approx 0 \ ,$$

or

$$\{\phi_\beta, H\} + \sum_\gamma \{\phi_\beta, \phi_\gamma\} v^\gamma \approx 0 \ ,$$

where $H \approx H_0$. Define the matrix

$$Q_{\beta\gamma} = \{\phi_\beta, \phi_\gamma\} \ ; \quad \beta, \gamma = 1, ..., n - R_W \ .$$

i) If the rank R_Q of this matrix is equal to $n - R_W$, then Q^{-1} exists, and the Lagrange-multipliers v^γ are completely determined on Γ_P. It is convenient however to define v^α *strongly* as follows:

$$v^\alpha(q,p) = -\sum_\beta Q^{-1}_{\alpha\beta}\{\phi_\beta, H\} \ .$$

Correspondingly we define the total Hamiltonian (3.20) by

$$H_T = H - \sum_\beta \phi_\alpha Q^{-1}_{\alpha\beta}\{\phi_\beta, H\} \ .$$

ii) If $R_Q < n - R_W$, then we can find R_Q linear combinations of the constraints,

$$\phi'_\beta = \sum_\gamma c_{\beta\gamma}\phi_\gamma$$

such that

$$\det\{\phi'_\alpha, \phi'_\beta\} \not\approx 0 \ ; \quad \alpha \ , \beta = 1, ..., R_Q$$

and

$$\{\phi'_\alpha, \phi'_\beta\} \approx 0 \ ; \quad \alpha = R_Q + 1, ..., n - R_W \ , \quad \forall\beta$$

where we have suitably ordered the constraints. Correspondingly only R_Q of the Lagrange multipliers are fixed via the persistence equations

$$\{\phi'_\beta, H\} + \sum_{\gamma=1}^{R_Q}\{\phi'_\beta, \phi'_\gamma\}v'^\gamma \approx 0 \ , \quad \beta = 1, ..., R_Q \ .$$

For the remaining constraints ϕ'_β, $\beta = R_Q + 1, ..., n - R_W$, their presistence in time demands that

$$\{\phi'_\beta, H_T\} \approx \{\phi'_\beta, H\} \approx 0 \ \ , \ \beta = R_Q + 1, \cdot\cdot\cdot, n - R_W \ .$$

This leads in general to *secondary constraints* which we denote by a small latin subscript,

$$\varphi_a = 0 \quad (\textit{secondary constraints}).$$

By repeating this procedure for the secondary constraints, one generates in general several "generations" of constraints. Note that the presistence equations for the constraints are always determined from the time derivative computed with the *total* Hamiltonian. A new constraint is generated from φ_a if i) the Poisson bracket of φ_a with all the primary constraints vanish on the subspace defined by *all* the constraints generated up to this point, and ii) $\{\varphi_a, H\}$ does not vanish on this subspace. In the case where i) does not hold, one is led to relations between Lagrange multipliers and canonical variables. [7] In the following we denote all constraints gained in this way collectively by $\Omega \equiv \{\Omega_A\}$:

$$\{\Omega_A\} = (\phi_\alpha, \varphi_a), \quad A = 1, \ldots, N \ .$$

3.4.1 Particular algorithm for generating the constraints

We now present a particular algorithm for generating the secondary constraints, which will prove very useful in later chapters. In this algorithm, the chain of

[7] The algorithm makes sure that we are never led to an overdetermined system of equations for the "velocities" v^β.

constraints generated from a given primary constraint ϕ_α will be denoted by $\phi_\ell^{(\alpha)}$, where ℓ labels the "level" in the chain, and the subscript $\ell = 0$ is reserved for the parent of the chain, i.e. the primary constraint $\phi_0^{(\alpha)} \equiv \phi_\alpha$.

Consider a given primary constraint $\phi_0^{(\alpha)} = 0$. This constraint must hold for all times, i.e.,

$$\{\phi_0^{(\alpha)}, H\} + \sum_\beta \{\phi_0^{(\alpha)}, \phi_0^{(\beta)}\} v^\beta \approx 0 \,.$$

Two possibilities arise:

1) The Poisson bracket of $\phi_0^{(\alpha)}$ with some of the primary constraints $\{\phi_0^{(\beta)}\}$ does not vanish weakly. In that case no new constraint arises from the requirement of persistence in time of $\phi_0^{(\alpha)} = 0$; instead a relation is required to hold among the v^β's.

2) The Poisson bracket of $\phi_0^{(\alpha)}$ with the entire set of primary constraints vanishes weakly. In this case we are led to the equation

$$\{\phi_0^{(\alpha)}, H\} \approx 0 \,.$$

Now, either this equation is satisfied identically, or a new constraint is generated. Suppose the latter is the case, then we denote the new constraint by $\phi_1^{(\alpha)}$, and define it through a strong equality:

$$\phi_1^{(\alpha)} \equiv \{\phi_0^{(\alpha)}, H\} \,.$$

Requiring the persistence in time of this (secondary) constraint, [8] we may (or may not) generate a new constraint, depending on whether the Poisson bracket of $\phi_1^{(\alpha)}$ with all the primary constraints vanish on the subspace defined by all the previous constraints, or not. Proceeding in this way it is evident that the constraints belonging to a given primary constraint $\phi_0^{(\alpha)}$ will satisfy the recursive relation

$$\phi_{\ell+1}^{(\alpha)} = \{\phi_\ell^{(\alpha)}, H\} \,. \tag{3.58}$$

No new constraint is generated once the rhs vanishes identically on the subspace defined by all the constraints up to the ℓ'th level. Note that we have defined the constraints (3.58) by a strong equality, rather than evaluating the Poisson brackets on the surface defined by all constraints up the ℓ'th level. At this point this is just a matter of simplifying the notation, but it will be convenient later on, when we discuss the local symmetries of a Lagrangian. In principle, however, only a weak equality is needed, where from now on *"weak equality" will always be understood as an equality holding on the subspace* Γ_Ω *defined*

[8] Recall that the time derivative is always computed as a Poisson bracket with H_T, subject to the constraints generated up to this level.

by all the constraints. This iterative procedure can be carried out “chain by chain”, or “level by level”.

Constructing in this way the descendents associated with each of the primary (parent) constraints, we are left at the end of the day with a set of primary and secondary constraints. Not all of these will however in general be linearly independent. In the following we will denote the independent set thus generated (including the primary constraints) by Ω_A $(A = 1, 2, \cdots)$.

3.5 First and second class constraints. Dirac brackets

Once we have extracted all the constraints of the theory, we are in the position of classifying them [Dirac 1964].

Definition: *A constraint is called “first class” if its Poisson bracket with all the constraints $\{\Omega_A\}$ vanishes weakly.*

First class constraints will be denoted by $\{\Omega^{(1)}_{A_1}\}$, where $A_1 = 1, \ldots, N_1$. Explicitly,

$$\{\Omega^{(1)}_{A_1}, \Omega_B\} \approx 0 \ , \quad A_1 = 1, \ldots, N^{(1)} \ ; \quad B = 1, \ldots, N \ . \tag{3.59}$$

The remaining constraints we label by $\Omega^{(2)}_{A_2}$ and are called *second class.* We shall assume that these constraints form an irreducible set; i.e., there exists no linear combination of the second class constraints which is first class. This is the case if

$$\det Q(q,p) \neq 0 \, ,$$

where $Q(q,p)$ is a matrix with elements

$$Q_{A_2,B_2} = \{\Omega^{(2)}_{A_2}, \Omega^{(2)}_{B_2}\} \, .$$

Indeed, if the determinant would vanish, then the homogeneous set of equations

$$\xi^{A_2}\{\Omega^{(2)}_{A_2}, \Omega^{(2)}_{B_2}\} \approx 0$$

would possess a non-trivial solution for the ξ^{A_2}, implying the existence of further first class constraints.

Conversely we have: a set of constraints $\{\psi_\ell\}$ is second class, if and only if the matrix

$$C_{k\ell} = \{\psi_k, \psi_\ell\}$$

is invertible on Γ ($\det C \neq 0$). For assume that there exists a linear combination $\zeta^\ell \psi_\ell$ such that it is first class. Then

$$\zeta^\ell \{\psi_\ell, \psi_k\} \approx 0$$

has a non-trivial solution ζ^ℓ. But then it follows that $det\{\psi_\ell, \psi\}$ must vanish weakly.

Finally we remark that since $Q_{A_2 B_2}(q,p)$ is an antisymmetric matrix, it follows that *the number of second class constraints is necessarily even*, since the determinant of an antisymmetric matrix in odd dimensions vanishes.

In what follows, it will be necessary to extend the concept of "first class constraints" to "first class functions" defined in phase space:

Definition: *A function $F(q,p)$ is first class, if its Poisson bracket with all the constraints vanishes weakly, i.e.,*

$$\{F, \Omega_A\} \approx 0\,, \quad for\ all \quad A\ .$$

This in turn implies that

$$\{F, \Omega_A\} = \sum_B f_{AB} \Omega_B\ ,$$

where $\{\Omega_A\}$ is the full set of first and second class constraints, and f_{AB} are in general functions of q and p. We now make the following assertion:

Assertion

> *The Poisson bracket of any two first-class functions of the canonical variables is first class.*

Indeed, let F and G be first class, i.e.,

$$\{F, \Omega_A\} = \sum_B f_{AB} \Omega_B\ ,$$

$$\{G, \Omega_A\} = \sum_B g_{AB} \Omega_B\ .$$

Since F, G and Ω_A are functions of the canonical variables, this is also the case for f and g. Making use of the Jacobi identity (3.23) one readily finds that

$$\{\{F, G\}, \Omega_A\} \approx \sum_B g_{AB}\{\Omega_B, F\} - \sum_B f_{AB}\{\Omega_B, G\} \approx 0 \ .$$

Hence $\{F, G\}$ is first class.

Dirac brackets

Having classified the constraints, we now re-examine the Hamilton equations of motion (3.19) and expose explicitly those degrees of freedom, which are associated with local symmetries of the theory. Thus in the examples discussed above we have seen that, depending on the model considered, some of the "Lagrange multipliers" v^α multiplying the primary constraints in (3.20) may, in certain cases, be actually determined as functions of the canonical variables, and therefore are not arbitrary.

The primary constraints can also be classified as being first and second class. We denote them by $\phi^{(1)}_{\alpha_1}$ and $\phi^{(2)}_{\alpha_2}$, respectively. Note that, as before, the Greek subscript is reserved to label the primary constraints. The total Hamiltonian (3.20) then takes the form

$$H_T = H + \sum_{\alpha_1} v^{(1)}_{\alpha_1}\phi^{(1)}_{\alpha_1} + \sum_{\alpha_2} v^{(2)}_{\alpha_2}\phi^{(2)}_{\alpha_2} \ .$$

Having revealed, following the Dirac algorithm, the complete set of constraints hidden in the Hamilton equations, we now make use of them in order to fix the "velocities" $v_2^{\alpha_2}$ associated with the second class primary constraints. Since by the Dirac algorithm all the constraints are time independent, it follows that they satisfy the following equations

$$\{\Omega^{(2)}_{A_2}, H\} + \sum_{\alpha_2} Q_{A_2\alpha_2} v^{(2)}_{\alpha_2} \approx 0 \, , \tag{3.60}$$

$$\{\Omega^{(1)}_{A_1}, H\} \approx 0 \ ,$$

where we have defined in a strong sense

$$Q_{A_2B_2} = \{\Omega^{(2)}_{A_2}, \Omega^{(2)}_{B_2}\} \ . \tag{3.61}$$

Equations (3.60) determine the "velocities" $v^{(2)}_{\alpha_2}$. Indeed, it follows from this equation that

$$\sum_{B_2} Q^{-1}_{A_2B_2}\{\Omega^{(2)}_{B_2}, H\} \approx -\sum_{\beta_2} \delta_{A_2\beta_2} v^{(2)}_{\beta_2} \ . \tag{3.62}$$

Setting $A_2 = \alpha_2$ we have that

$$v^{(2)}_{\alpha_2} = -\sum_{B_2} Q^{-1}_{\alpha_2 B_2}\{\Omega^{(2)}_{B_2}, H\}\ ,$$

which defines the "velocities" $v^{(2)}_{\alpha_2}$, associated with the second class primary constraints, as functions of the canonical variables. Note that we have again defined the velocities by a strong equation, rather than as a weak equality. Clearly the classical equations of motion are insensitive to this, but it will turn out to be relevant for defining the corresponding quantum theory.

Let us summarize our findings so far. The Lagrange multipliers associated with second class primaries are determined by the persistence conditions as functions of the *canonical variables.* This is not the case for the Lagrange multipliers associated with the first class primaries. In fact, as we will see, they remain undetermined, and are associated with gauge degrees of freedom of the system.

In principle we can now write down the total Hamiltonian which will only involve the velocities associated with the first class primary constraints as undetermined parameters. A particular symmetric form can however be obtained by observing that upon setting $A_2 \to a_2$ in (3.62), with the small latin index labeling the secondary *second* class constraints, we are led to the equation

$$\sum_{B_2} Q^{-1}_{a_2 B_2}\{\Omega^{(2)}_{B_2}, H\} \approx 0\,. \tag{3.63}$$

Note that this equation is automatically ensured by our algorithm. Making use of this equation, as well as of the expression for $v^{(2)}_{\alpha_2}$ above, we can also write the total Hamiltonian in the form

$$H_T = H^{(1)} + \sum_{\alpha_1} v^{(1)}_{\alpha_1}\Omega^{(1)}_{\alpha_1}\ , \tag{3.64}$$

with

$$H^{(1)} = H - \sum_{A_2,B_2} \Omega^{(2)}_{A_2} Q^{-1}_{A_2 B_2}\{\Omega^{(2)}_{B_2}, H\}\ , \tag{3.65}$$

where the superscript (1) on $H^{(1)}$ indicates that it is first class. We now prove that this is indeed the case. To this effect we notice that by the definition of $H^{(1)}$, we have that

$$\{\Omega^{(2)}_{A_2}, H^{(1)}\} \approx 0\,. \tag{3.66}$$

Furthermore, since the time derivative of all constraints vanishes weakly, as implied by the Dirac algorithm, we have in particular

$$0 \approx \{\Omega^{(1)}_{A_1}, H_T\} \approx \{\Omega^{(1)}_{A_1}, H^{(1)}\}\,.$$

Hence $H^{(1)}$ has also a weakly vanishing Poisson bracket with all first class constraints. From here and (3.66) it follows that $H^{(1)}$ is first class.

Relations (3.64) and (3.65) suggest the introduction of the following (Dirac) bracket of two functions f and g of the canonical variables:

$$\{f, g\}_D \equiv \{f, g\} - \sum_{A_2, B_2} \{f, \Omega^{(2)}_{A_2}\} Q^{-1}_{A_2 B_2} \{\Omega^{(2)}_{B_2}, g\} \,. \tag{3.67}$$

Important properties of the Dirac bracket are:

i) The algebraic properties of Dirac brackets are the same as those of the Poisson brackets.

ii) The Dirac bracket of any function of the canonical variables with a second class constraint vanishes strongly, i.e.,

$$\{F, \Omega^{(2)}_{A_2}\}_D \equiv 0 \,. \tag{3.68}$$

Because of i) this implies that the second class constraints can be implemented strongly *within* the Dirac bracket. Making use of the definition (3.67) and of (3.64) we see that for an arbitrary function of q and p the equations of motion (3.19) can be written in the form

$$\dot{f} \approx \{f, H\}_D + \sum_{\alpha_1} v^{(1)}_{\alpha_1} \{f, \phi^{(1)}_{\alpha_1}\} \,. \tag{3.69}$$

For a purely second class system the last term on the rhs is absent. On the other hand, for a purely first class system the Dirac bracket coincides with the Poisson bracket.

Let us reconsider our example 1 in this formulation. The constraints (3.50) and (3.51) are second class. The matrix $Q_{A_2 B_2}$, defined in (3.61), and its inverse read

$$\mathbf{Q} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \,, \qquad \mathbf{Q^{-1}} = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \,.$$

From the definition of the Dirac bracket (3.67) one readily verifies that

$$\{y, p_x\}_D = \frac{1}{2} \,, \quad \{x, y\}_D = \frac{1}{2} \,.$$

All other Dirac brackets are given by the corresponding Poisson brackets, i.e., they have the canonical form. One readily checks that the Dirac bracket of all the constraints vanish identically, and that the equations of motion (3.47)-(3.50) are reproduced by (3.69) with $\phi^{(1)}_{\alpha_1} = p_y$, and f standing generically for x, y, p_x, p_y.

Chapter 4

Symplectic Approach to Constrained Systems

4.1 Introduction

With every second order Lagrangian $L(q, \dot{q})$ one can associate a "symplectic" Lagrangian $L_s(Q, \dot{Q})$ which is first order in $\dot{Q}_a$, and whose Euler-Lagrange equations of motion coincide with those derived from $L(q, \dot{q})$ after an appropriate identification of variables. In the case of an unconstrained system, such a first order formulation can be obtained immediately. Thus the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L_s}{\partial \dot{Q}_\alpha} - \frac{\partial L_s}{\partial Q_\alpha} = 0$$

associated with the (symplectic) first order Lagrangian [1]

$$L_s(Q, \dot{Q}) = \sum_{i=1}^{n} p_i \dot{q}_i - H_c(q, p) \ , \quad Q_\alpha := (q_i, p_i) \ ; \quad \alpha = 1, \cdots, 2n \tag{4.1}$$

yield the Hamilton equations of motion associated with the corresponding *second* order Lagrangian $L(q, \dot{q})$ with canonical Hamiltonian $H_c(q, p)$. Furthermore, the dynamics described by the above Lagrangian is fully equivalent to that described by

$$L_s = \frac{1}{2}\sum_{i=1}^{n}(p_i \dot{q}_i - \dot{p}_i q_i) - H_c(q, p) \ , \tag{4.2}$$

[1] We use everywhere lower indices, at the expense of explicitly introducing summation signs for summation over repeated indices.

where we have dropped an irrelevant total derivative. L_s can be written in the so-called symplectic form [Faddeev 1988]

$$L_s = \frac{1}{2}\sum_{\alpha,\beta=1}^{2n} Q_\alpha \bar{f}_{\alpha\beta}\dot{Q}_\beta - H_0(Q)\,, \tag{4.3}$$

where the matrix $\bar{f}_{\alpha\beta}$ has the following off-diagonal structure with the ordering $(\vec{q},\vec{p})$:

$$\bar{f}_{\alpha\beta} := \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} .$$

Alternatively, (4.3) is of the form

$$L_s = \sum_\beta A_\beta(Q)\dot{Q}_\beta - V_s(Q)\ , \tag{4.4}$$

where

$$A_\beta = \frac{1}{2}\sum_\alpha Q_\alpha \bar{f}_{\alpha\beta}\ , \tag{4.5}$$

with $V_s(Q) = H_0(Q)$. Consider now (4.4), where we leave A_α unspecified. Noting that

$$\frac{\partial L_s}{\partial Q_\alpha} = \sum_\beta \frac{\partial A_\beta}{\partial Q_\alpha}\dot{Q}_\beta - \frac{\partial V_s}{\partial Q_\alpha}\ ,$$
$$\frac{d}{dt}\frac{\partial L_s}{\partial \dot{Q}_\alpha} = \sum_\beta \frac{\partial A_\alpha}{\partial Q_\beta}\dot{Q}_\beta\ ,$$

the equations of motion associated with (4.4) read

$$\sum_b f_{\alpha\beta}\dot{Q}_\beta = K^{(0)}_\alpha\ , \tag{4.6}$$

where

$$f_{\alpha\beta} = \frac{\partial A_\beta}{\partial Q_\alpha} - \frac{\partial A_\alpha}{\partial Q_\beta} \tag{4.7}$$

and

$$K^{(0)}_\alpha = \frac{\partial V_s}{\partial Q_\alpha}\ . \tag{4.8}$$

In the case where L_s is given by (4.2), $f_{\alpha\beta} = \bar{f}_{\alpha\beta}$. Since for a non-singular system the inverse of $f_{\alpha\beta}$ exists (f has no zero modes), we can solve (4.6) for the velocities. Next let us define the following generalized Poisson bracket [Faddeev 1988]

$$\{Q_\alpha, Q_\beta\} \equiv f^{-1}_{\alpha\beta}\ .$$

This definition can be generalized to arbitrary functions of the symplectic variables as follows:

$$\{F, G\} = \sum_{\alpha,\beta} \frac{\partial F}{\partial Q_\alpha} f^{-1}_{\alpha\beta} \frac{\partial G}{\partial Q_\beta} \ .$$

With this definition the equations of motion (4.6) can be put into a Hamiltonian form,

$$\dot{Q}_\alpha = \sum_b f^{-1}_{\alpha\beta} \frac{\partial V_s}{\partial Q_\beta} = \{Q_\alpha, V_s(Q)\} \, ,$$

with the symplectic potential V_s playing the role of the Hamiltonian. [2]

As an example consider the (non-singular) second order Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - V(q) \ . \tag{4.9}$$

Introducing the variable $p = \dot{q}$, the equation of motion,

$$\ddot{q} = -\frac{\partial V}{\partial q}$$

can be rewritten as

$$\dot{q} = p, \quad \dot{p} = -\frac{\partial V}{\partial q} \ .$$

These equations are the Euler-Lagrange equations of motion following from the (symplectic) Lagrangian

$$L_s(Q, \dot{Q}) = p\dot{q} - V_s(p, q) \ , \quad V_s(p, q) = \frac{1}{2}p^2 + V(q) \, .$$

We see that V_s is just the canonical Hamiltonian associated with the *second order* Lagrangian (4.9). Thus the Euler-Lagrange equations derived from L_s are just the Hamilton euations of motion following from (4.9). In the notation of (4.4) we have $Q_1 = q,\ Q_2 = p,\ A_1 = p,\ A_2 = 0$, and correspondingly $f_{ab} = -\epsilon_{ab}$. f is a non-singular matrix with the inverse $f^{-1}_{\alpha\beta} = \epsilon_{\alpha\beta}$. Correspondingly we have the canonical Poisson brackets $\{q, p\} = 1, \quad \{q, q\} = \{p, p\} = 0$. With the identification $V_s(q, p) = H_c(q, p)$, equations (4.6) take the form of Hamilton equations of motion:

$$\dot{q} = \{q, V_s\} = \frac{\partial V_s}{\partial p} \quad , \quad \dot{p} = \{p, V_s\} = -\frac{\partial V_s}{\partial q} \, .$$

[2] Note that the momenta conjugate to $\{Q_\alpha\}$ do not appear. In fact, from (4.4) one sees that they are constrained to be functions of Q: $P_\alpha - A_\alpha(Q) = 0$. These are primary constraints, equal in number with the "coordinates" $Q := (q, p)$. It is for this reason that we labeled the coordinates Q_α with a Greek index.

4.2 The case f_{ab} singular

In the case of a singular Lagrangian, the equivalent symplectic Lagrangian can still be written in the form (4.4), but the matrix $f_{\alpha\beta}$ in (4.7) is singular, i.e., not invertible. That L_s can still be written in the form (4.4) can be seen by starting from the *total Lagrangian*,

$$L_T = \sum_{i=1}^{n} p_i \dot{q}_i - H_0(q,p) - \sum_{\bar{\alpha}=1}^{n_0} v^{\bar{\alpha}} \phi_{\bar{\alpha}} \ , \tag{4.10}$$

where $\phi_{\bar{\alpha}} = 0$ are the primary constraints. The Euler derivative of $L_T(Q, \dot{Q})$ yields the correct Hamilton equations of motion. Since the v^{α}'s play the role of auxiliary variables, we can use the equations (primary constraints)

$$\frac{\partial L_T}{\partial v^{\bar{\alpha}}} = \phi_{\bar{\alpha}}(q,p) = 0$$

to eliminate some of the q's and p's, in the Lagrangian L_T. The resulting new Lagrangian will then be of the form (4.4), but with a reduced number of degrees of freedom. This form will actually be the starting point for the Faddeev-Jackiw approach [Faddeev 1988] to constrained systems, which we shall discuss in the last section of this chapter.

Having established that the symplectic Lagrangian will also be of the form (4.4) in the case of a singular system, but with a singular matrix (4.7), let us now proceed with the solution of the equations of motion.

Let r_0 be the rank of the $2n \times 2n$ matrix $f^{\alpha\beta}$. Then there exist $2n - r_0$ (zero-level) zero modes of f, which - following a similar notation as that used in chapter 2 - we denote by $\vec{u}^{(0,k)}$ $(k = 1, \cdots, 2n - r_0)$. Multiplying equations (4.6) from the left with these zero modes, we are led to the zero-level Lagrangian constraints

$$\sum_{\alpha=1}^{2n} u_{\alpha}^{(0,k)} K_{\alpha}^{(0)} = 0 \, , \quad k = 1, 2, \cdots, 2n - r_0 \, .$$

These constraints only depend on Q. Some of them may vanish identically. The remaining ones we denote by $\varphi^{(0,a_0)}$ $(a_0 = 1, 2, \cdots, N_0)$. [3] The corresponding zero modes $\vec{u}^{(0,a_0)}$ we refer to as "non-trivial" level-zero, zero modes.

In general there are further constraints hidden in equations (4.6). In order to unravel them, we implement the conservation of $\varphi^{(0,a_0)} = 0$,

$$\left(\frac{\partial \varphi^{(0,a_0)}}{\partial Q_{\alpha}} \right) \dot{Q}_{\alpha} = 0 \ , \quad a_0 = 1, 2, \cdots, N_0 \ ,$$

[3] Note that we have denoted the constraints by φ in anticipation of the fact that, within a Hamiltonian formulation, they correspond to secondary constraints associated with the second order Lagrangian.

and adjoin these equations to (4.6). This leads to the following enlarged set of equations

$$\sum_\beta W^{(1)}_{A_1\beta}(Q)\dot{Q}_\beta = K^{(1)}_{A_1}(Q)\,, \tag{4.11}$$

where $W^{(1)}_{A_1\beta}$ are now the elements of a rectangular matrix [4]

$$W^{(1)}_{A_1\beta} := \begin{pmatrix} f_{\alpha\beta} \\ M^{(0)}_{a_0\beta} \end{pmatrix} \;, \tag{4.12}$$

with

$$M^{(0)}_{a_0\beta} = \frac{\partial\varphi^{(0,a_0)}}{\partial Q_\beta} \;,$$

$$K^{(1)}_{A_1} := \begin{pmatrix} \vec{K}^{(0)} \\ \vec{0} \end{pmatrix} \;,$$

and

$$\vec{K}^{(0)} = \vec{\nabla}V(Q)\,.$$

Here $\vec{0}$ is an N_0-component vector. We now look for zero modes $\vec{u}^{(1,a_1)}$ ($a_1 = 1, 2, \cdots, N_1$) of $W^{(1)}$, leading to possibly new constraints, and repeat the above steps, adjoining the time derivative of the new constraints to the equations of motion (4.11). Repeating this algorithm, the iterative process terminates after L steps, when no new constraints are generated.

Denote the full set of constraints $\{\varphi^{(0,a_0)}\}$, $\{\varphi^{(1,a_1)}\}$ $\cdots$ generated by the algorithm by φ_a, $a = 1, \cdots, N$, and the set (α, a) by the collective index A. The *final* set of equations of motion can then be written in the form

$$\sum_\beta W^{(L)}_{A\beta}\dot{Q}_\beta = K^{(L)}_A \tag{4.13}$$

where

$$W^{(L)}_{A\beta} := \begin{pmatrix} f_{\alpha\beta} \\ M_{a\beta} \end{pmatrix} \quad ; \quad L \geq 1, \tag{4.14}$$

with

$$M_{a\beta} = \frac{\partial\varphi_a}{\partial Q_\beta} \tag{4.15}$$

and

$$K^{(L)}_A := \begin{pmatrix} \vec{K}^{(0)} \\ \vec{0} \end{pmatrix} \,. \tag{4.16}$$

[4] The upper entry stands for a $2n \times 2n$ matrix, while the lower entry is an $N_0 \times 2n$ matrix.

Denoting by $\vec{u}^{(L,a)}(Q)$ the complete set of left zero-modes of the matrix $W^{(L)}_{A\beta}(Q)$, the constraints are given by

$$\varphi_a := \vec{u}^{(L,a)} \cdot \vec{K}^{(L)} = 0 \ .$$

Equations (4.13) represent $2n + N$ equations for the $2n$ velocities $\{\dot{Q}_\alpha\}$. In general such a set of equations would be overdetermined and admit no non-trivial solution. Since the additional N equations were however generated by a self-contained algorithm from the original Euler-Lagrange equations, equations (4.13) do in fact admit a non-trivial solution.

In the following we consider the case where the first order Lagrangian (4.4) describes a purely second class system in the Dirac terminology. In that case we make the following assertion [Rothe 2003b]:

Assertion

For a second class system the unique solution to (4.13) for the velocities is given by

$$\dot{Q}_\alpha = \sum_\beta F^{-1}_{ab} K^{(0)}_\beta \tag{4.17}$$

where F^{-1} is the inverse of the matrix F obtained by extending the rectangular matrix W defined in (4.14) to the antisymmetric square matrix

$$F_{AB} := \begin{pmatrix} f_{\alpha\beta} & -M^T_{\alpha b} \\ M_{a\beta} & \mathbf{0} \end{pmatrix} \ , \tag{4.18}$$

with $M_{a\beta}$ defined in (4.15), and $M^T_{\alpha b} = M_{b\alpha}$.

Proof of Assertion

Consider an enlarged space ξ on which the square matrix (4.18) is to act (we streamline the notation in a self-evident way),

$$\xi_A := (Q_\alpha, \rho_a) \ ,$$

and the following equations:

$$\sum_B F_{AB}\dot{\xi}_B = K^{(L)}_A \ . \tag{4.19}$$

As we shall prove further below, $\det F \neq 0$ for a second class system. We can then solve these equations for the velocities $\dot{\xi}_B$:

$$\dot{\xi}_A = \sum_B F^{-1}_{AB} K^{(L)}_B \ . \tag{4.20}$$

Let us write the inverse F^{-1} of the (antisymmetric) matrix (4.18) in the form

$$F_{AB}^{-1} := \begin{pmatrix} \tilde{f}_{\alpha\beta} & -\tilde{M}_{\alpha b}^{T} \\ \tilde{M}_{a\beta} & \omega_{ab} \end{pmatrix} . \tag{4.21}$$

Then $F^{-1}F = 1$ implies,

$$\begin{aligned} &\tilde{f}_{\alpha\gamma} f_{\gamma\beta} - \tilde{M}_{\alpha c}^{T} M_{c\beta} = \delta_{\alpha\beta} \ , \\ &\tilde{f}_{\alpha\gamma} M_{\gamma b}^{T} = 0 \ , \\ &\tilde{M}_{a\gamma} f_{\gamma\beta} + \omega_{ac} M_{c\beta} = 0 \, , & (4.22) \\ &\tilde{M}_{a\gamma} M_{\gamma b}^{T} = -\delta_{ab} \ , & (4.23) \end{aligned}$$

where repeated indices are summed.

Consider eqs. (4.19) which, written out explicitly, read

$$\sum_{\beta} f_{\alpha\beta} \dot{Q}_{\beta} - \sum_{b} M_{\alpha b}^{T} \dot{\rho}_{b} = K_{\alpha}^{(0)} \ , \tag{4.24}$$

$$\sum_{\beta} M_{a\beta} \dot{Q}_{\beta} = 0 \ .$$

From the definition of $M_{a\beta}$ in (4.15) we see that the last equation states that $\dot{\varphi}_a = 0$, where $\varphi_a = 0$ are the constraints hidden in the Lagrange equations of motion. From (4.6) it follows that the second term on the lhs of (4.24) must vanish. Making use of (4.23) this in turn implies that $\dot{\rho}_a = 0$ for all $a, (a = 1, \cdots, N)$. Setting $\dot{\rho}_a = 0$ in (4.20), we have from (4.21) and (4.16) that

$$\dot{Q}_{\alpha} = \sum_{\beta} \tilde{f}_{\alpha\beta} K_{\beta}^{(0)} \ , \tag{4.25}$$

$$0 = \sum_{\beta} \tilde{M}_{a\beta} K_{\beta}^{(0)} \ . \tag{4.26}$$

Equation (4.25) is nothing but (4.17), and (4.26) is just the statement that $\varphi_a = 0$ (within a Hamiltonian formulation these correspond to secondary constraints). To see this we notice that according to (4.22), the vectors

$$\vec{v}_{C}^{(a)} := (\tilde{M}_{a\gamma}, \omega_{ac})$$

are the left zero modes of the matrix (4.14). Hence

$$\varphi_a \equiv \sum_{A} v_{A}^{(a)} K_{A}^{(L)} = \sum_{\beta} \tilde{M}_{a\beta} K_{\beta}^{(0)} \ .$$

This concludes the proof.

4.2.1 Example: particle on a hypersphere

In the following we first illustrate the above considerations in terms of a simple example. Consider the following Lagrangian, which is referred to in the literature as describing the non-linear sigma model in quantum mechanics,

$$L(\vec{q}, \dot{\vec{q}}, \lambda) = \frac{1}{2}\dot{\vec{q}}^2 + \lambda(\vec{q}^2 - 1) \ , \tag{4.27}$$

where $\vec{q} = (q_1, \cdots, q_n)$. The equivalent symplectic Lagrangian reads,

$$L_s = \vec{p} \cdot \dot{\vec{q}} - V_s(q, p, \lambda) \tag{4.28}$$

with

$$V_s = \frac{1}{2}\vec{p}^2 - \lambda(\vec{q}^2 - 1) \ .$$

With the identifications $Q_\alpha := (\vec{q}, \vec{p}, \lambda)$ and $A_\alpha := (\vec{p}, \vec{0}, 0)$ in (4.4), the equations of motion are given by (4.6), with

$$f_{\alpha\beta} := \begin{pmatrix} \mathbf{0} & -\mathbf{1} & \vec{0} \\ \mathbf{1} & \mathbf{0} & \vec{0} \\ \vec{0}^T & \vec{0}^T & 0 \end{pmatrix}$$

and

$$K_\alpha^{(0)} := \begin{pmatrix} -2\lambda\vec{q} \\ \vec{p} \\ -(\vec{q}^2 - 1) \end{pmatrix} \ .$$

The matrix f has one "zero-level" zero mode,

$$\vec{u}^{(0,1)} := (\vec{0}, \vec{0}, 1) \ ,$$

implying the constraint

$$\varphi^{(0,1)} = -\sum_a u_a^{(0,1)} K_a^{(0)} = \vec{q}^2 - 1 = 0 \ .$$

Adding the time derivative of this constraint to the equation of motion (4.6) we are led to (4.12) with

$$W^{(1)} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} & \vec{0} \\ \mathbf{1} & \mathbf{0} & \vec{0} \\ \vec{0}^T & \vec{0}^T & 0 \\ 2\vec{q}^T & \vec{0}^T & 0 \end{pmatrix} ,$$

which is seen to possess two "level-one" zero modes:

$$\vec{u}^{(1,1)} = (\vec{0}, 2\vec{q}, 0, -1) \ , \quad \vec{u}^{(1,2)} = (\vec{0}, \vec{0}, 1, 0) \, .$$

The second zero-mode just reproduces the previous constraint, while the first zero mode implies a new constraint

$$\varphi^{(1,1)} = \vec{u}^{(1,1)} \cdot \vec{K}^{(1)} = 2\vec{p} \cdot \vec{q} = 0\,.$$

Incorporating the time derivative of this constraint into our algorithm, leads to the level 2 matrix

$$W^{(2)} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} & \vec{0} \\ \mathbf{1} & \mathbf{0} & \vec{0} \\ \vec{0}^T & \vec{0}^T & 0 \\ 2\vec{q}^T & \vec{0}^T & 0 \\ 2\vec{p}^T & 2\vec{q}^T & 0 \end{pmatrix} ,$$

which is seen to possess the three zero modes,

$$\vec{u}^{(2,1)} = \begin{pmatrix} \vec{0} \\ \vec{0} \\ 1 \\ 0 \\ 0 \end{pmatrix} , \quad \vec{u}^{(2,2)} = \begin{pmatrix} 0 \\ 2\vec{q} \\ 0 \\ -1 \\ 0 \end{pmatrix} , \quad \vec{u}^{(2,3)} = \begin{pmatrix} 2\vec{q} \\ -2\vec{p} \\ 0 \\ 0 \\ 1 \end{pmatrix} .$$

In addition to the constraints $\varphi^{(0,1)} = 0$, $\varphi^{(1,1)} = 0$, these imply a new constraint

$$\varphi^{(2,1)} := 2\lambda\vec{q}^2 + \vec{p}^2 = 0 \;. \tag{4.29}$$

This constraint fixes the so far arbitrary (Lagrange multiplier) λ as a function of $\vec{q}$ and $\vec{p}$. Hence we are taken to a third level with the corresponding enlarged matrix given by

$$W^{(3)} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} & \vec{0} \\ \mathbf{1} & \mathbf{0} & \vec{0} \\ \vec{0}^T & \vec{0}^T & 0 \\ 2\vec{q}^T & \vec{0}^T & 0 \\ 2\vec{p}^T & 2\vec{q}^T & 0 \\ 4\lambda\vec{q}^T & 2\vec{p}^T & 2\vec{q}^2 \end{pmatrix} .$$

As one readily checks, $W^{(3)}$ has no *new* zero modes. Hence the algorithm terminates at this point, and the equation of motions (4.13) takes the form

$$\sum_\beta W^{(3)}_{A\beta}\dot{Q}_\beta = K^{(3)}_A \;.$$

Proceeding as described above, we extend the matrix $W^{(3)}$ to a square matrix F. This results in the invertible matrix,

$$F = \begin{pmatrix} \mathbf{0} & -\mathbf{1} & \vec{0} & -2\vec{q} & -2\vec{p} & -4\lambda\vec{q} \\ \mathbf{1} & \mathbf{0} & \vec{0} & \vec{0} & -2\vec{q} & -2\vec{p} \\ \vec{0}^T & \vec{0}^T & 0 & 0 & 0 & -2\vec{q}^2 \\ 2\vec{q}^T & \vec{0}^T & 0 & 0 & 0 & 0 \\ 2\vec{p}^T & 2\vec{q}^T & 0 & 0 & 0 & 0 \\ 4\lambda\vec{q}^T & 2\vec{p}^T & 2\vec{q}^2 & 0 & 0 & 0 \end{pmatrix} .$$

As we shall demonstrate in the following section, the elements of this matrix are just $\{\Omega_A, \Omega_B\}$, where Ω_A are the full set of primary and secondary constraints associated with the symplectic Lagrangian (4.28). The coordinates q_i, momenta p_i and λ are then given by solving eq. (4.17) or (4.25).

4.3 Interpretation of $W^{(L)}$ and F

In the above Lagrangian approach the concept of a primary constraint did not appear. On the other hand, from the Dirac point of view, the symplectic Lagrangian (4.4) describes a system with a *primary* constraint for each of the $2n$ coordinates Q_α,

$$\phi_\alpha(Q,P) := P_\alpha - A_\alpha(Q) = 0\,, \quad \alpha = 1, \cdots, 2n\ , \tag{4.30}$$

where P_α is the momentum canonically conjugate to Q_α. Since the Lagrangian (4.4) is first order in the time derivatives, the corresponding Hamiltonian on Γ_P, H_0, is just given by the potential $V_s(Q)$,

$$H_0 = V_s(Q)\ , \tag{4.31}$$

and hence does not depend on the momenta. The dependence on the momenta enters only in the *total* Hamiltonian via the *primary* constraints:

$$H_T(Q,P) = V_s(Q) + v^\alpha \phi_\alpha(Q,P)\,. \tag{4.32}$$

As a consequence we have that the Dirac algorithm applied to this symplectic formulation will only generate the first level of secondary constraints, and comes to a stop at this point. Let us look at this in more detail.

Persistence of the primary constraints will in general lead to secondary constraints, which we label as before by an index a_0: $\varphi_{a_0} = 0$. It is easy to show that they are identical with the constraints generated at the lowest level of the Lagrangian algorithm. Indeed, consider the persistence equations for the primary constraints ϕ_α:

$$\{\phi_\alpha, H_T\} \approx \{\phi_\alpha, V_s\} + \{\phi_\alpha, \phi_\beta\} v^\beta = 0\ . \tag{4.33}$$

From (4.30) and the definition of $f_{\alpha\beta}$ in (4.7) we see that

$$\{\phi_\alpha, \phi_\beta\} = f_{\alpha\beta}(Q) \ . \tag{4.34}$$

Hence equations (4.33) read,

$$f_{\alpha\beta} v^\beta = K_\alpha^{(0)} \ , \tag{4.35}$$

where $K_\alpha^{(0)}$ has been defined in (4.8). Multiplying (4.35) with the (level-zero) left-zero modes $u^{(0,k)}$ of f corresponds in the Dirac language to searching for linear combinations of the primary constraints which have weakly vanishing Poisson brackets with *all* the primary constraints. This is evident from (4.34). These left zero modes will depend only on the Q_a's. Correspondingly we have for the independent (level-zero) secondary constraints

$$\varphi^{(0,a_0)}(Q) = \sum_\beta u_\beta^{(0,a_0)} \frac{\partial V_s}{\partial Q_\beta} \ ,$$

which are now only functions of the Q_α's, and thus have vanishing Poisson brackets with H_0. Hence their time persistence leads to

$$\sum_\beta v^\beta \{\varphi^{(0,a_0)}, \phi_\beta\} = \sum_\beta M_{a_0\beta}^{(0)} v^\beta = 0 \ ,$$

where $M_{a_0\beta}^{(0)} = \frac{\partial\varphi^{(0,a_0)}}{\partial Q_\beta}$. Adjoining these equations to (4.35) we have that

$$W_{A_1\beta}^{(1)} v^\beta = K_{A_1}^{(1)} \ , \tag{4.36}$$

where A_1 stands for the set (α, a_0) and $W^{(1)}$ is given by (4.12). Upon noting that

$$\dot{Q}_\alpha = \{Q_\alpha, H_T\} = v^\alpha \tag{4.37}$$

we are led to (4.11). Since the secondary constraints $\varphi^{(0,a_0)}$ and the potential V_s only depend on the Q_α's, the Dirac algorithm for the first order Lagrangian (4.4) terminates, for the first order Lagrangian (4.4), at this point. To extract the remaining information embodied in the Lagrangian algorithm discussed in section 2, we must now proceed in a way completely analogous to the Lagrangian algorithm. Thus by taking appropriate linear combinations of equations (4.36), new constraints may be generated which are again functions of only the Q_α's. This just corresponds to looking for left-zero modes of $W^{(1)}$. The new constraints are thus identical with those derived in the Lagrangian approach at Lagrangian level "one". Proceeeding in this way we of course recover all the constraints generated by the Lagrangian algorithm, and are finally led to eq. (4.13).

The matrix (4.14) can also be rewritten in terms of Poisson brackets as follows

$$W^{(L)}_{A\beta} := \begin{pmatrix} \{\phi_\alpha, \phi_\beta\} \\ \{\varphi_a, \phi_\beta\} \end{pmatrix} \ ,$$

where φ_a now denote the complete set of secondary constraints. This follows immediately from the definition (4.15) and the structure of the primary constraints (4.30). Furthermore, the extended matrix F_{AB} defined in (4.18) can also be written in terms of Poisson brackets. Thus consider the extended Hamiltonian

$$H_E = H + v^\alpha \phi_\alpha + \lambda^a \varphi_a \ , \tag{4.38}$$

where H is given by (4.31). Next consider the persistence equation for the primary constraints,

$$\{\phi_\alpha, v^\beta \phi_\beta + \lambda^b \varphi_b\} - \frac{\partial V_s}{\partial Q_\alpha} \approx 0 \ ,$$

and the persistence equation for the secondary constraints (which only depend on Q),

$$\{\varphi_a, v^\beta \phi_\beta + \lambda^b \varphi_b\} \approx 0 \ .$$

They can be combined to the weak equations

$$\begin{pmatrix} f_{\alpha\beta} & \{\phi_\alpha, \varphi_b\} \\ \{\varphi_a, \phi_\beta\} & \{\varphi_a, \varphi_b\} \end{pmatrix} \begin{pmatrix} v^\beta \\ \lambda^b \end{pmatrix} \approx \begin{pmatrix} \frac{\partial V_s}{\partial Q_\alpha} \\ 0 \end{pmatrix} \ .$$

Upon noting that $\{\varphi_a, \varphi_b\} = 0$, since the secondary constraints only depend on the Q_α's, and making use of $\{\varphi_a, \phi_\beta\} = (\partial\varphi_a/\partial Q_\beta) = M_{a\beta}$, as well as of (4.37), we are led to equation (4.19), with λ^b playing the role of ρ_b. These equations can also be written in the compact form

$$\{\Omega_A, \Omega_B\}\xi^B = K^{(0)}_A \ ,$$

where $\Omega_A := (\phi_\alpha, \varphi_a)$. If the Ω_A's form a second class system, then the matrix F_{AB} in (4.18) is invertible, as we had claimed before. Hence the extension of the rectangular matrix (4.12) to the square matrix (4.18) corresponds to working with the extended Hamiltonian (4.38), rather than with the total Hamiltonian (4.32).

4.4 The Faddeev-Jackiw reduction

In the Dirac approach of dealing with singular systems, the dynamical equations involve the variables of the entire phase space, including also unphysical gauge degrees of freedom. This remains true even after fixing a gauge and introducing Dirac brackets. Observables are then restricted by the requirement that they "commute" with all the first class constraints. A systematic algorithm for arriving at a set of Hamilton equations of motion involving only the physical (unconstrained) degrees of freedom has been proposed by Faddeev and Jackiw (FJ) [Faddeev 1988]. That this algorithm is equivalent to the Dirac procedure has been shown in [García 1997]. Here we shall only summarize the basic steps of the Faddeev-Jackiw approach and discuss a simple example.

Starting from a second order Lagrangian $L(q, \dot{q})$ describing a constrained system, the objective is to arrive at a first order Lagrangian, possessing a canonical form of the type (4.1), where the coordinates q_i and momenta p_i ($i = 1, \cdots, n$) are replaced by a *reduced* set of canonically conjugate variables q^*_n and p^*_n ($n = 1, \cdots, n^*$) which are completely unconstrained, and whose dynamics is described by the standard Hamilton equations of motion with a reduced Hamiltonian $H^*(q^*, p^*)$. This is accomplished by the Faddeev-Jackiw algorithm:

i) By Darboux's theorem [5] it is always possible to perform a transformation from the q_i, p_i ($i = 1, \cdots, n$) to a new set of variables $q'_i, p'_i, Z_{\underline{a}}$ ($i = 1, \cdots, n'$; $\underline{a} = 1, \cdots, 2n - 2n'$) with (q'_i, p'_i) canonically conjugate variables, so that (4.1) takes the form

$$L'_s = \sum_{i=1}^{n'} p'_i \dot{q}'_i - U(q', p', Z) . \tag{4.39}$$

Note that the $Z_{\underline{a}}$'s do not possess a canonically conjugate partner and thus play the role of auxiliary variables.

ii) Next make use of the Euler-Lagrange equations of motion for the auxiliary variables,

$$\frac{\partial L'_s}{\partial Z_{\underline{a}}} = 0 \ ,$$

to express as many as possible of the $Z_{\underline{a}}$'s in (4.39) in terms of the q'_i's, p'_i's, and the remaining Z's, which we denote by $\lambda_{\bar{a}}$. This is allowed since the Z's play the role of auxiliary variables. One is then led to a Lagrangian having the form [6]

$$L'_s = \sum_{i=1}^{n'} p'_i \dot{q}'_i - H(q', p') - \sum_{\bar{a}} \lambda_{\bar{a}} \Omega_{\bar{a}}(q', p') \ . \tag{4.40}$$

Note that the $\lambda_{\bar{a}}$'s now appear linearly. This must be the case since we have assumed that we eliminated the maximum number of $Z_{\underline{a}}$'s. At this point we

[5] See [Jackiw 1993] for a simplified derivation.

[6] We continue to label the symplectic Lagrangian with L'_s although its functional form has changed.

see that the last term on the rhs of (4.40) implies the constraints $\Omega_{\bar{a}}(q', p') = 0$ among the canonically conjugate variables. These are true Lagrangian constraints of the theory defined by L.

iii) Next we implement the constraints via the equation of motion

$$\frac{\partial L'_s}{\partial \lambda_{\bar{a}}} = 0 \; .$$

Elimination of the dependent variables using these constraints will in general lead to a non-canonical form for the new Lagrangian. After performing another Darboux transformation, one is then again left with an expression of the form (4.39), but with a reduced set of canonical pairs (q''_i, p''_i), $i = 1, \ldots, n''$, and a reduced set of new auxiliary Z-fields:

$$L"_s = \sum_{i=1}^{n''} p''_i \dot{q}''_i - U(q'', p'', Z') \, .$$

One now repeats the above steps over and over again until one is left with an expression involving no longer any auxiliary fields and having the canonical form,

$$L^*_s = \sum_n p^*_n \dot{q}^*_n - H^*(q^*, p^*) \; .$$

Here the algorithm stops.

Clearly the practical application of this algorithm may become quite involved, especially because of the Darboux transformations. Other algorithms have been presented subsequently in the literature [Barcelos-Neto 1992, Montani 1993], which avoid the in general technically difficult Darboux transformation, but may fail under certain conditions [Rothe 2003b].

We conclude this chapter by presenting a simple example involving only one Darboux transformation.

Example

Consider the second order Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - ax\dot{y} + \frac{b}{2}(x-y)^2 \, . \tag{4.41}$$

The following steps represent a useful guide for initiating the reduction process.

There is one primary constraint

$$\phi = p_y + ax = 0 \; . \tag{4.42}$$

The canonical Hamiltonian on the primary surface thus reads,

$$H_0 = \frac{1}{2}p_x^2 - \frac{b}{2}(x-y)^2\,.$$

A convenient starting point for constructing the symplectic Lagrangian now proceeds from the "*total*" Lagrangian

$$L_T = p_x\dot{x} + p_y\dot{y} - H_0 - \lambda\phi\ ,$$

where ϕ is the primary constraint (4.42). Eliminating p_y via the primary constraint by setting $p_y = -ax$ in the expression for L_T, yields the new Lagrangian

$$L_s = (p_x + ay)\dot{x} - \frac{1}{2}p_x^2 + \frac{b}{2}(x-y)^2\ ,$$

where a total derivative has been dropped. We now perform a (trivial) Darboux transformation by setting

$$p_1 = p_x + ay\ , \quad q_1 = x\,. \tag{4.43}$$

We are then left with

$$L'_s = p_1\dot{q}_1 - \frac{1}{2}(p_1 - ay)^2 + \frac{b}{2}(q_1 - y)^2\,. \tag{4.44}$$

The Lagrangian L'_s is of the form (4.39) with y playing the role of an auxiliary variable Z. We now make use of the equation of motion

$$\frac{\partial L'_s}{\partial y} = 0$$

to eliminate y in favor of (q_1, p_1),

$$y = \frac{ap_1 - bq_1}{\gamma}\ , \tag{4.45}$$

where

$$\gamma = a^2 - b\ .$$

Introducing this expression in (4.44), we obtain the following reduced Lagrangian which only involves the unconstrained variables q_1 and p_1, and has the canonical form

$$L_{FJ} = \dot{q}_1 p_1 - H^*(q_1, p_1)$$

with the reduced Hamiltonian

$$H^*(q_1, p_1) = -\frac{b}{2\gamma}(p_1 - aq_1)^2\ . \tag{4.46}$$

At this point the FJ reduction process terminates, and the quantization can proceed in the standard way for an unconstrained system.

In terms of the original variables the constraint (4.45) reads

$$ap_x - b(x-y) = 0 \ , \tag{4.47}$$

which we recognize to be the secondary constraint as obtained from the original Hamiltonian associated with the second order Lagrangian (4.41). The Poisson bracket of the primary constraint (4.42) with the secondary constraint (4.47) is given by $a^2 - b \equiv \gamma$. Hence for $\gamma \neq 0$ the system is second class and the transformation (4.45) is well defined, while the singularity at $\gamma = 0$ reflects the fact that the constraints become first class for $\gamma = 0$.

Using (4.43) and (4.45) in order to eliminate q_1 and p_1 in favor of x and y, it is a straightforward matter to verify that the Hamilton equations of motion associated with the reduced Hamiltonian (4.46) are equivalent to the second order Lagrange equations of motion associated with the Lagrangian (4.41).

Although we have found it helpful to start from the total Lagrangian, the FJ-approach does not require the notion of primary and secondary constraints, nor their classification into first and second class. We should remark however that such a reduction process may be difficult to carry out since it requires knowledge of the Darboux transformation. Thus one may have to resort after all to Dirac's method in order to circumvent this problem.

Chapter 5

Local Symmetries within the Dirac Formalism

5.1 Introduction

The Lagrangian approach to local symmetries described in chapter 2 has the merit of yielding directly the transformation laws for the coordinates in terms of an *independent* set of time dependent parameters, whose number equals the number of independent gauge identities generated by the Lagrangian algorithm. The fact that local symmetries are intimately tied to the existence of first class constraints in the Hamiltonian formalism is not manifest in this approach. In this chapter we consider the problem of unravelling the local symmetries of a theory from a purely Hamiltonian point of view. This is a necessary prerequisite for quantizing the theory.

In the following we shall consider purely first class systems. We will therefore denote the primary constraints $\phi^{(1)}_{\alpha_1}$, and the complete set of constraints $\Omega^{(1)}_{A_1}$ simply by ϕ_α and Ω_A, respectively. Furthermore, the first class Hamiltonian $H^{(1)}$ defined in (3.65) coincides with the Hamiltonian H, weakly equivalent to the canonical Hamiltonian H_0 evaluated on the primary surface. The generalization to the case of mixed systems is straightforward and will be reserved for a comment at the end of this chapter.

5.2 Local symmetries and canonical transformations

The physical state of a system at any time t should be uniquely determined by specifying $q(t)$ and $p(t)$. [1] If however $q(t)$ and $p(t)$ are not uniquely determined by the initial values $q(t_0)$ and $p(t_0)$, this implies a condition on a function $f(q,p)$ to be an observable of the theory.

Consider the time evolution of a general function $F(q(t),p(t))$, which in the following we simply denote by $f(t)$. We then have [2]

$$\begin{aligned} f(t_0+\delta t) &= f(t_0) + \dot{f}(t_0)\delta t + \mathcal{O}((\delta t)^2) \\ &\approx f(t_0) + \{f(t_0), H_T\}\delta t + \mathcal{O}((\delta t)^2) \\ &\approx f(t_0) + \{f(t_0), H\}\delta t + v^\beta\{f(t_0),\phi_\beta\}\delta t + \mathcal{O}((\delta t)^2)\ , \end{aligned}$$

where $f(t_0) = F(q(t_0),p(t_0))$, with $(q(t_0),p(t_0))$ a point located on the primary surface. For another choice of the parameters v^β $(v^\beta \to v'^\beta)$, $f(t_0)$ will evolve as follows

$$f'(t_0+\delta t) \approx f(t_0) + \{f(t_0), H\}\delta t + v'^\beta\{f(t_0),\phi_\beta\}\delta t + \mathcal{O}((\delta t)^2).$$

The difference δf between the two results is given to order $O(\delta t)$ by

$$\delta f \approx \delta\rho^\beta\{f(t_0),\phi_\beta\} + \mathcal{O}((\delta t)^2)\ , \tag{5.1}$$

where $\delta\rho^\beta = \delta t(v'-v)^\beta$. Hence ϕ_β are generators of infinitesimal (local) *point-transformations* with infinitesimal parameters $\delta t(v'-v)^\beta$. Observables $O(q,p)$ should be invariant under such transformations mediating between the two trajectories. For this to be the case we must have

$$\{O,\phi_\beta\} \approx 0\ , \quad \forall\beta.$$

Consider now two consecutive infinitesimal point-transformations

$$f \xrightarrow{\varepsilon^\alpha\phi_\alpha} f' \xrightarrow{\gamma^\beta\phi_\beta} f''\ ,$$

and compare this with

$$f \xrightarrow{\gamma^\beta\phi_\beta} \tilde{f}' \xrightarrow{\varepsilon^\alpha\phi_\alpha} \tilde{f}''\ .$$

[1] In the following q and p denote collectively the complete set of coordinates and canonical momenta. Furthermore, we shall make extensive use of the Einstein summation convention.

[2] Poisson brackets are always understood to be evaluated at equal times.

Explicitly we have

$$\begin{aligned} f'' &\approx f' + \gamma^\alpha \{f', \phi_\alpha\} \\ &\approx f + \varepsilon^\alpha \{f, \phi_\alpha\} + \gamma^\alpha \{f + \varepsilon^\beta \{f, \phi_\beta\}, \phi_\alpha\} \ , \end{aligned}$$

with a corresponding expression for $\tilde{f}''$ with ε and γ interchanged. For the difference $\tilde{f}'' - f''$ we then obtain

$$\begin{aligned} \tilde{f}'' - f'' &\approx \varepsilon^\alpha \gamma^\beta \big(\{\{f, \phi_\beta\}, \phi_\alpha\} - \{\{f, \phi_\alpha\}, \phi_\beta\} \big) + \eta^\alpha \{f, \phi_\alpha\} \\ &\approx \varepsilon^\alpha \gamma^\beta \{f, \{\phi_\beta, \phi_\alpha\}\} + \eta^\alpha \{f, \phi_\alpha\} \ , \end{aligned} \tag{5.2}$$

where

$$\eta^\alpha = \epsilon^\beta \{\gamma^\alpha, \phi_\beta\} - \gamma^\beta \{\epsilon^\alpha, \phi_\beta\} \ .$$

The difference (5.2) of two such successive transformations should again leave physics invariant. Now, by assumption, the ϕ_α belong to a first class system $\{\Omega_A\}$ in the sense of Dirac. Hence

$$\{\phi_\alpha, \phi_\beta\} = U^D_{\alpha\beta} \Omega_D \ .$$

But (5.2) can only be written in form (5.1), provided the Poisson algebra of the first class *primary* constraints closes on itself: $\{\phi_\alpha, \phi_\beta\} = U^\gamma_{\alpha\beta} \phi_\gamma$. Since this is in general not the case, this indicates, that in fact *all* first class constraints - primary and secondary - need to be included as generators of local point transformations, and that observables must be invariant under transformations generated by all the first class constraints:

$$\{\mathcal{O}(q,p), \Omega_A\} \approx 0 \quad (\mathcal{O} : \textit{observable}) \ . \tag{5.3}$$

This is known as *Dirac's conjecture*, and will be the subject of chapter 6. Accordingly, we are led to consider symmetry transformations in phase space of the form

$$q^i(t) \to q^i(t) + \delta q^i(t) \ , \quad p_i(t) \to p_i(t) + \delta p_i(t) \ ,$$

where

$$\delta q^i(t) = \epsilon^A(t) \{q^i(t), \Omega_A\} \ , \quad \delta p_i(t) = \epsilon^A(t) \{p_i(t), \Omega_A\} \ , \tag{5.4}$$

$(A = 1, \cdots, N_1)$ with N_1 the number of first class constraints. The parameter-functions ϵ^A can in principle depend implicitly on time through the coordinates and momenta, as well as explicitly on time.

The above transformations are transformations in the unconstrained phase space. But physics is restricted to take place at any time on the surface defined by the constraints. In fact, if (q, p) is a point located on the constrained surface

$\Omega_A(q,p) = 0$ $(A = 1, \cdots N_1)$, then this is also true for $(q+\delta q, p+\delta p)$. Indeed, with (5.4) we have,

$$\Omega_A(q+\delta q, p+\delta p) = \Omega_A(q,p) + \epsilon^B\{\Omega_A, \Omega_B\} \approx 0 \ ,$$

since the constraints are first class, and $\Omega_A(q,p)$ vanishes for all A.

For a general function $f(q,p)$, eq. (5.4) implies that

$$\delta f(q,p) = \epsilon^A(t)\{f(q,p), \Omega_A\} \ . \tag{5.5}$$

In the following we now show that the functions ϵ^A cannot all be chosen arbitrarily.

5.3 Local symmetries of the Hamilton equations of motion

Consider the Hamilton equations of motion (3.22), i.e.,

$$\begin{aligned} \dot{q}^i &= \{q^i, H\} + v^\alpha\{q^i, \phi_\alpha\} \ , \\ \dot{p}_i &= \{p_i, H\} + v^\alpha\{p_i, \phi_\alpha\} \ , \\ \Omega_A(q,p) &= 0 \quad , \ \forall A \ . \end{aligned} \tag{5.6}$$

We have written these equations as strong equalities, where it is always understood that the Poisson brackets must be calculated before imposing the constraints. Note that we have made use of our freedom to add explicitly also the secondary constraints, since they are implied by the consistency of the equations of motion (3.22).

Equations (5.6) involve the arbitrary "velocities" $v^\alpha(t)$, which we expect to be associated with gauge degrees of freedom. Suppose that for a given set of functions $v^\alpha(t)$ the solution to the equations of motion are given by $q^i(t)$ and $p_i(t)$. We then look for new solutions differing by $\delta q^i(t)$ and $\delta p_i(t)$, with the $v^\alpha(t)$'s replaced by $v'^\alpha(t) = v^\alpha(t) + \delta v^\alpha(t)$. If the $v^\alpha(t)$'s are indeed gauge degrees of freedom, then physical observables must take the same values on both trajectories. Define

$$\begin{aligned} F^i(q,p) &= \{q^i, H\} \quad ; \quad G^i_\alpha(q,p) = \{q^i, \phi_\alpha\}, \\ \tilde{F}_i(q,p) &= \{p_i, H\} \quad ; \quad \tilde{G}_{i\alpha}(q,p) = \{p_i, \phi_\alpha\} \ . \end{aligned}$$

Consider further the following infinitesimal variations δq^i and δp_i, induced by the first class constraints Ω_A,

$$\delta q^i(t) = \epsilon^A(t)\{q^i(t), \Omega_A\} = \epsilon^A(t)\left(\frac{\partial \Omega_A}{\partial p_i(t)}\right) \ ,$$

$$\delta p_i(t) = \epsilon^A(t)\{p_i(t), \Omega_A\} = -\epsilon^A(t)\left(\frac{\partial \Omega_A}{\partial q^i(t)}\right) \ ,$$

where $\Omega_A = \Omega_A(q(t), p(t))$, and the ϵ^A's are (so far still arbitrary) infinitesimal functions of time. The above equations can also be written in the (weak) form

$$\delta q^i \approx \{q_i(t), G(t)\} \quad , \qquad \delta p_i \approx \{p_i(t), G(t)\} \ ,$$

where

$$G = \epsilon^A(t)\Omega_A(t) \tag{5.7}$$

is the generator of the transformations induced by the first class constraints.

We now demand that if q^i and p_i are a solution to (5.6), then $q^i + \delta q^i$ and $p_i + \delta p_i$ will be solution to [3]

$$\frac{d}{dt}(q^i + \delta q^i) = F^i(q + \delta q, p + \delta p) + v'^\alpha G^i_\alpha(q + \delta q, p + \delta p) \ ,$$
$$\frac{d}{dt}(p_i + \delta p_i) = \tilde{F}_i(q + \delta q, p + \delta p) + v'^\alpha \tilde{G}_{i\alpha}(q + \delta q, p + \delta p) \ ,$$

$$\Omega_A(q + \delta q, p + \delta p) = 0 \ ,$$

for some suitably chosen v'^α. Note that we have allowed v'^α to differ from v^α, consistent with our requirement of form invariance. Making use of the equations of motion for $q^i(t)$ and $p_i(t)$, and of $\Omega_A(q(t), p(t)) = 0$, we are led to the conditions

$$\frac{d}{dt}\delta q^i = \delta F^i + v^\alpha \delta G^i_\alpha + \delta v^\alpha G^i_\alpha \ , \tag{5.8}$$
$$\frac{d}{dt}\delta p_i = \delta \tilde{F}_i + v^\alpha \delta \tilde{G}_{i\alpha} + \delta v^\alpha \tilde{G}_{i\alpha} \ , \tag{5.9}$$

$$\delta\Omega_A = 0 \ , \tag{5.10}$$

where $\delta v^\alpha = v'^\alpha - v^\alpha$, and

$$\delta F^i = \epsilon^B(t)\{\{q^i, H\}, \Omega_B\} \ , \quad \delta G^i_\alpha = \epsilon^B(t)\{\{q^i, \phi_\alpha\}, \Omega_B\} \ , \tag{5.11}$$

$$\delta\Omega_A = \epsilon^B(t)\{\Omega_A, \Omega_B\} \ .$$

Similar expressions hold for $\delta\tilde{F}^i$ and $\delta\tilde{G}_{i\alpha}$ with q^i replaced by p_i. Note that, as already remarked, the condition (5.10) is automatically fulfilled, since the constraints Ω_A are first class, and $q^i(t)$ and $p_i(t)$ satisfy $\Omega_A(q(t), p(t)) = 0$.

Now, from (5.4) it follows that

$$\frac{d}{dt}\delta q^i = \dot{\epsilon}^A(t)\{q^i, \Omega_A\} + \epsilon^A(t)\frac{d}{dt}\{q^i, \Omega_A\} \ ,$$

[3] In the following we suppress the time-argument of the phase space variables and Lagrange multipliers.

where, making use of the equations of motion

$$\frac{d}{dt}\{q^i,\Omega_A\} \approx \{\{q^i,\Omega_A\},H\} + v^\alpha(t)\{\{q^i,\Omega_A\},\phi_\alpha\} \ .$$

Corresponding expressions hold for $\frac{d}{dt}\delta p_i$ and $\frac{d}{dt}\{p_i,\Omega_A\}$. Inserting the expressions for $\frac{d}{dt}\delta q^i$, $\frac{d}{dt}\delta p_i$, δF^i, δG^i_α, $\delta\tilde{F}_i$ and $\delta\tilde{G}_{i\alpha}$ into (5.8) and (5.9), and making use of the Jacobi identity, one is led to the following *on-shell* conditions on ϵ^A and δv^α,

$$[\dot{\epsilon}^A - \epsilon^B(V_B^{\ A} + v^\beta U^A_{\beta B})]\{q^i,\Omega_A\} - \delta v^\alpha\{q^i,\phi_\alpha\} \approx 0 \ ,$$

where $V_B^{\ A}(q,p)$ and $U^A_{\beta B}(q,p)$ are defined through

$$\{\Omega_A,\Omega_B\} = U^D_{AB}\Omega_D \ , \tag{5.12}$$

$$\{H,\Omega_A\} = V_A^{\ B}\Omega_B \ . \tag{5.13}$$

From here we conclude that the Hamilton equations of motion remain form invariant under *on-shell* transformations of the form (5.4), if [Banerjee 1999]

$$\dot{\epsilon}^a \approx \epsilon^B(V_B^{\ a} + v^\beta U^a_{\beta B}) \ , \tag{5.14}$$

$$\delta v^\alpha \approx \dot{\epsilon}^\alpha - \epsilon^B(V_B^{\ \alpha} + v^\beta U^\alpha_{\beta B}) \,, \tag{5.15}$$

where the small latin index a labels the first class *secondary* constraints. Hence the number of independent functions $\epsilon^B(t)$ equals the number of *primary* constraints. Note that (5.14) and (5.15) are *on-shell* equations, and that no specification has been made as to how the first class constraints have been chosen! On shell means that the structure functions $V_B^{\ A}(q,p)$, $U^A_{\beta B}(q,p)$ are evaluated for $q^i(t)$ and $p_i(t)$ satisfying the Hamilton equations of motion on the constrained surface for a given set of functions $v^\alpha(t)$.

5.4 Local symmetries of the total and extended action

In the previous section we have studied the local symmetries of the Hamilton equations of motion, generated by the first class constraints. The so obtained relations among the ϵ^A parameters, the Lagrange multipliers and the variations δv^α must also follow by studying the local symmetries of the *total* action, since its variation leads to the Hamilton equations of motion (5.6).

i) Local symmetries of the total action

Consider the total action

$$S_T = \int dt \left(p_i \dot{q}^i - H_T\right) \ ,$$

where

$$H_T = H + v^\beta \phi_\beta \ .$$

For a general variation $p_i \to p_i + \delta p_i, \ q^i \to q^i + \delta q^i$ we have

$$\begin{aligned}\delta(p\dot{q}) &= p\frac{d}{dt}(\delta q) + \delta p \frac{dq}{dt} \\ &= \frac{d}{dt}(p\delta q) - \dot{p}\delta q + \delta p \dot{q} \ ,\end{aligned}$$

where for simplicity we have dropped the (summation) index i on the coordinates and momenta. In particular, for transformations of the form (5.4) we obtain

$$\begin{aligned}\delta(p\dot{q}) &= \frac{d}{dt}(p\delta q) - \epsilon^A\{q, \Omega_A\}\dot{p} + \epsilon^A\{p, \Omega_A\}\dot{q} \\ &= \frac{d}{dt}\left[\epsilon^A\left(p\frac{\partial\Omega_A}{\partial p} - \Omega_A\right)\right] + \dot{\epsilon}^A\Omega_A \ . \end{aligned} \qquad (5.16)$$

Consider next δH_T. Since H is first class, [4] it follows from (5.5), making use of (5.12) and (5.13), that

$$\delta H_T = \epsilon^B (V_B^A \Omega_A + v^\beta U_{\beta B}^A \Omega_A) + \delta v^\alpha \phi_\alpha \ ,$$

or [5]

$$\delta S_T = \int dt \left[\dot{\epsilon}^A \Omega_A - \epsilon^B (V_B^A + v^\beta U_{\beta B}^A)\Omega_A - \delta v^\alpha \phi_\alpha\right] \ .$$

Note that we have not made use of the equations of motion. Requiring that $\delta S_T = 0$ thus leads again to (5.14) and (5.15), but this time off-shell.

ii) Local symmetries of the extended action

The dynamics of *observables*, i.e., quantities whose Poisson brackets with the first class constraints vanish weakly, is equally well described by the equations

[4] Recall again that we are discussing a purely first class system.

[5] We assume that the boundary term arising from the total time derivative does not contribute, i.e., that the ϵ^A's vanish at the boundary.

of motion (3.22), or by replacing the total Hamiltonian in (3.22) by the *extended Hamiltonian*

$$H_E = H + \xi^A \Omega_A \ . \tag{5.17}$$

Hence all first class constraints are now treated on the same level. It must however be remembered that the extended Hamiltonian can only be defined *after* having generated all the constraints via the Dirac algorithm, using the *total* Hamiltonian. The variation with respect to the q^i's, p_i's and the ξ^A's of the extended action

$$S_E = \int dt\ (\dot{q}^i p_i - H_E)$$

leads to the full set of Hamilton equations of motion including all the constraints.

Since observables are insensitive to the Lagrange parameters ξ^A in (5.17), any transformation of the coordinates and momenta that can be absorbed by these parameters will leave observables invariant. We therefore expect that the number of arbitrary functions parametrizing the generator in (5.7) will now be equal to the total number of first class constraints. In fact, consider the variation induced by a gauge transformation of the form (5.4). One readily verifies that

$$\delta S_E = \int dt[\dot{\epsilon}^B \Omega_B - \epsilon^A (V_A^B + \xi^C U_{CA}^B)\Omega_B - \delta\xi^B \Omega_B]\,,$$

so that the requirement of gauge invariance now leads to

$$\delta\xi^B = \dot{\epsilon}^B - \epsilon^A (V_A^B + \xi^C U_{CA}^B)\,.$$

Thus there are no longer any restrictions on the parameters $\{\epsilon^A\}$, which are now arbitrary functions of $q(t)$, $p(t)$ and t.

Example

Consider once more the Lagrangian (3.52):

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x-y)^2\,.$$

The corresponding extended Hamiltonian (5.17) reads

$$H_E = \frac{1}{2}p_x^2 - (p_x - x)y - \frac{1}{2}x^2 + \xi^1\phi + \xi^2\varphi\ ,$$

where

$$\phi := p_y = 0\,, \quad \varphi := p_x - x = 0$$

are the primary and secondary (first class) constraints. The corresponding equations of motion take the form,

$$\dot{x} = \{x, H_E\} = p_x - y + \xi^2,$$
$$\dot{y} = \{y, H_E\} = \xi^1 \ ,$$

and the generator of gauge transformations (5.7) reads

$$G = \varepsilon^1 p_y + \varepsilon^2 (p_x - x) \ .$$

Taking ϵ^2 to be only a function of time, we have

$$\delta x = \{x, G\} = \varepsilon^2; \quad \delta y = \varepsilon^1; \quad \delta p_x = \varepsilon^2; \quad \delta p_y = 0.$$

The structure functions defined in (5.12) and (5.13) are given by $U^A_{BC} = 0$ and

$$V_A^B := \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \ ,$$

so that

$$\delta\xi^1 = \dot{\varepsilon}^1, \ \delta\xi^2 = \dot{\varepsilon}^2 + \varepsilon^1 - \varepsilon^2 \, .$$

The above transformations are symmetry transformations of the Hamilton equations of motion generated by H_E.

5.5 Local symmetries of the Lagrangian action

In the previous section we have studied the (off-shell) local symmetries of the total and extended action. We now show, that the corresponding Lagrangian action

$$S = \int dt \ L(q, \dot{q}) \tag{5.18}$$

is also invariant under a similar set of gauge transformations, but with the Lagrange multipliers v^α identified with the (arbitrary) velocities $\dot{q}^\alpha$. To show this we first rewrite the Lagrangian $L(q, \dot{q})$ in terms of the coordinates q^i, the independent momenta

$$p_a = \frac{\partial L}{\partial \dot{q}^a} \ ,$$

and the velocities $\dot{q}^\alpha$. The $\dot{q}^a$'s are then functions of q^i, p_a and $\dot{q}^\alpha$ (see (3.6)):

$$\dot{q}^a = f^a(q, \{p_b\}, \{\dot{q}^\alpha\}) \ . \tag{5.19}$$

Expressed in these variables the Lagrangian $L(q, \dot{q})$ is given by

$$L(q, \{f^a\}, \{\dot{q}^\alpha\}) = \tilde{L}(q, \{p_a\}, \{\dot{q}^\alpha\}) \ .$$

Consider the canonical Hamiltonian evaluated on the primary surface $p_\alpha - g_\alpha(q, \{p_a\}) = 0$:

$$H_0(q, \{p_a\}) = p_a f^a + g_\alpha \dot{q}^\alpha - \tilde{L} \ .$$

The action (5.18) then takes the form

$$S = \int dt \ \left(p_a f^a + g_\alpha \dot{q}^\alpha - H_0\right) \ ,$$

where the integrand is now a function of $\{q^i\}$, $\{p_a\}$ and $\{\dot{q}^\alpha\}$. Consider a variation of the action,

$$\delta S = \int dt \ [\delta(p_a f^a) + \delta(g_\alpha \dot{q}^\alpha) - \delta H_0] \ . \tag{5.20}$$

The first term appearing in the integrand can also be written in the form

$$\delta(p_a f^a) = f^a \delta p_a + \frac{d}{dt}(p_a \delta q^a) - \dot{p}_a \delta q^a \, , \tag{5.21}$$

where $\frac{d}{dt}\delta q^a$ has been identified with δf^a, as follows from (5.19) and $\frac{d}{dt}\delta q = \delta \dot{q}$. An analogous expression holds for the second term:

$$\delta(g_\alpha \dot{q}^\alpha) = \dot{q}^\alpha \delta g_\alpha + \frac{d}{dt}(g_\alpha \delta q^\alpha) - \dot{g}_\alpha \delta q^\alpha \ . \tag{5.22}$$

Next consider the variation (5.5) of g_α. With $\phi_\alpha \equiv p_\alpha - g_\alpha(q, p_a)$, we have that

$$\begin{aligned} \delta g_\alpha &= \epsilon^A \{p_\alpha, \Omega_A\} - \epsilon^A \{\phi_\alpha, \Omega_A\} \ , \\ &= -\epsilon^A \frac{\partial \Omega_A}{\partial q^\alpha} - \epsilon^A \{\phi_\alpha, \Omega_A\} \ . \end{aligned}$$

Furthermore

$$\delta q^\alpha = \epsilon^A \left(\frac{\partial \Omega_A}{\partial p_\alpha}\right) \ .$$

Making further use of

$$\left(\dot{q}^i \frac{\partial \Omega_A}{\partial q^i} + \dot{p}_i \frac{\partial \Omega_A}{\partial p_i}\right) = \frac{d}{dt}\Omega_A \, ,$$

we have from (5.21) and (5.22) that

$$\delta(p_a f^a + g_\alpha \dot{q}^\alpha) = \frac{d}{dt}\left[\left(p_i \frac{\partial \Omega_A}{\partial p_i} - \Omega_A\right)\epsilon^A\right] + \dot{\epsilon}^a \Omega_a - \epsilon^A \dot{q}^\alpha \{\phi_\alpha, \Omega_A\}. \tag{5.23}$$

The total derivative does not contribute to the action, since it is assumed that ϵ^A vanishes at the endpoints of the integration.

Finally, consider the variation δH_0:

$$\delta H_0 = \epsilon^A \{H_0, \Omega_A\} = \epsilon^A V_A^B \Omega_B \ ,$$

where we have made use of (5.13) with $H = H_0$, and of the fact that, because we are considering a purely first class system, H_0 is itself first class. With (5.23) one then finds that (5.20) vanishes on the surface of the *primary* constraints if the conditions

$$\dot{\epsilon}^a = \epsilon^B (V_B^a + \dot{q}^\beta U_{\beta B}^a) \tag{5.24}$$

are satisfied. This is nothing but the condition (5.14) with v^α identified with the undetermined velocity $\dot{q}^\alpha$, and with ${V_B}^a$ and $U_{\beta B}^a$ evaluated on the primary surface.

Example

The Lagrangian

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}(y - z) + \frac{1}{2}(x - y + z)^2$$

leads to two primary constraints,

$$\Omega_1 := p_y = 0 \quad \Omega_2 := p_z = 0 \,.$$

The canonical Hamiltonian reads

$$H_0 = \frac{1}{2}p_x^2 - (y - z)p_x - \frac{1}{2}x^2 + x(y - z) \ .$$

From Ω_1 and Ω_2 we generate just one secondary constraint,

$$\Omega_3 = \{H_0, \Omega_1\} = -\{H_0, \Omega_2\} = x - p_x \ .$$

This exhausts all secondary constraints. All the constraints are first class. Hence we expect the Lagrangian to possess a local symmetry. One symmetry is obvious: L is invariant under $x \to x,\ y \to y+\epsilon,\ z \to z+\epsilon$. But the Lagrangian is actually invariant under a larger set of infinitesimal transformations, as we now show.

For the structure functions U_{AB}^D and V_B^A defined in (5.12) and (5.13) we have ${V_1}^3 = -{V_2}^3 = {V_3}^3 = 1$ and $U_{AB}^D = 0$. Recalling that ϵ^1 and ϵ^2 are the parameters associated with the primary constraints, the recursion relation (5.24) then leads to the following equation for the parameter ϵ^3 associated with the secondary constraint $\Omega_3 = 0$:

$$\dot{\epsilon}^3 - \epsilon^1 + \epsilon^2 - \epsilon^3 = 0 \,.$$

Since there are two primary first class constraints, the solution involves two parameters, which we choose to be $\epsilon^3 = \alpha(t)$ and $\epsilon^2 = \beta(t)$. One is thus led to the solution

$$\epsilon^1 = \dot{\alpha} - \alpha + \beta \,.$$

The corresponding symmetry-transformations (5.4) of the coordinates are then given by,

$$\begin{aligned} \delta x &= -\alpha(t) \ , \\ \delta y &= \dot{\alpha} - \alpha + \beta \ , \\ \delta z &= \beta \,. \end{aligned}$$

As expected, one finds that the variation of the Lagrangian induced by these transformations is just a total derivative,

$$\delta L = -\frac{d}{dt}(\alpha x) \,,$$

so that $\delta S = 0$. Note that upon making the substitutions $\alpha \to -\beta$, $\beta \to \alpha$, the above set of transformations coincide with (2.36).

5.6 Solution of the recursive relations

In the following we shall seek a solution to (5.24) under the following assumptions [Banerjee 2000a]:

i) The Poisson bracket of *any* constraint with the M primary constraints is a linear combination of only the primary constraints. This implies $U^a_{\beta B} = 0$, and hence the absence of the last term in (5.24).

ii) The structure functions ${V_B}^a$ are either constants, or may be arbitrary functions of the phase space variables, provided that there are no "tertiary" constraints. That is, the Dirac algorithm terminates at the first level. Important bosonic examples are the Maxwell and Yang-Mills theories in the absence of matter sources.

In order to solve equations (5.24) with these restrictions it is convenient to organize the constraints into "families", where the parent of each family, labeled by "α", is given by a primary constraint $\phi_0^{(\alpha)}$, and the remaining members $\phi_i^{(\alpha)}$, $i = 1, 2 \cdots$, are recursively derived from [Shirzad 1999, Banerjee 2000a]

$$\{H, \phi_{i-1}^{(\alpha)}\} = \phi_i^{(\alpha)} \ , \quad i = 1, ..., N_\alpha \,. \tag{5.25}$$

The (not necessarily independent) set of constraints generated this way is then given by $\phi_i^{(\alpha)}$ $(\alpha = 1 \cdots M;\ i = 0 \cdots N_\alpha)$. With the above notation for the constraints, the structure functions V_A^B in (5.24) are correspondingly replaced by $V_{ij}^{\alpha\beta}$, and have, for $i < N_\alpha$, the simple form

$$V_{ij}^{\alpha\beta} = \delta^{\alpha\beta}\delta_{i,j-1} \ , \quad i = 0, \cdots, N_\alpha - 1 \ . \tag{5.26}$$

In this notation

$$\{H, \phi_i^{(\alpha)}\} = \sum_{j,\beta} V_{ij}^{\alpha\beta}\phi_j^{(\beta)} \ . \tag{5.27}$$

Notice that $V_{N_\alpha j}^{\alpha\beta}$ remains undetermined by our algorithm. In order to ensure that the constraints thus obtained are irreducible, we must adopt some systematic procedure. A possibility is to implement the Dirac algorithm level by level, decending from all primary constraints simultaneously. Scanning one by one through every member at each level, we terminate a family "α", if at a given level N_α, the Poisson bracket of the constraint $\phi_{N_\alpha}^{(\alpha)}$ with H can be written as a linear combination of all the other constraints obtained up to this point. This ensures the irreducibility of the constraints thus obtained. Organizing the families in this particular way then implies that $V_{N_\alpha j}^{\alpha\beta} = 0$ for $j > inf\{N_\alpha, N_\beta\}$. But whatever the procedure one may adopt for obtaining the irreducible set of constraints, the Poisson bracket of the final member of each family with H is given by,

$$\{H, \phi_{N_\alpha}^{(\alpha)}\} = \sum_{\beta=1}^{M} \sum_{j=0}^{inf(N_\alpha,N_\beta)} V_{N_\alpha j}^{\alpha\beta}\phi_j^{(\beta)} \ ,$$

where M is the number of primary constraints. Correspondingly equation (5.24) now reads [Banerjee 2000a]

$$0 = \frac{d\epsilon_i^{(\alpha)}}{dt} - \sum_{\beta=1}^{M}\sum_{j=0}^{N_\beta} \epsilon_j^{(\beta)} V_{ji}^{\beta\alpha} \ , \quad i = 1, \cdots, N_\alpha \ . \tag{5.28}$$

Let us choose as independent parameter-functions those associated with the last member in each family, and let them be arbitrary functions of time:

$$\alpha^\beta := \epsilon_{N_\beta}^{(\beta)}(t) \ . \tag{5.29}$$

Making use of (5.26), equations (5.28) then take the form

$$\frac{d\epsilon_i^{(\alpha)}}{dt} - \epsilon_{i-1}^{(\alpha)} - \sum_{\beta=1}^{M} \alpha^\beta V_{N_\beta i}^{\beta\alpha} = 0 \ , \quad i = 1, \cdots, N_\alpha \ . \tag{5.30}$$

The solution to this set of equations can be constructed iteratively, by starting with the last member of a family:

$$\epsilon_{N_\beta - 1}^{(\beta)} = \frac{d\alpha^\beta}{dt} - \sum_{\gamma=1}^{M} \alpha^\gamma V_{N_\gamma N_\beta}^{\gamma\beta} \ . \tag{5.31}$$

Continuing in this fashion, one easily sees that the general solution can be written in the form

$$\epsilon_i^{(\alpha)} = \sum_{n=0}^{N_\alpha - i} \sum_{\beta=1}^{M} \frac{d^n \alpha^\beta}{dt^n} A_{i(n)}^{\beta\alpha} \quad , \; i = 0, \cdots, N_\alpha \; , \tag{5.32}$$

with the normalization

$$A_{N_\alpha(0)}^{\beta\alpha} = \delta^{\beta\alpha} \; , \tag{5.33}$$

following from our choice of parametrization (5.29). Substituting (5.32) into (5.30) and comparing powers in the time derivatives, one is led to the recursion relations

$$A_{i(n-1)}^{\beta\alpha} = A_{i-1(n)}^{\beta\alpha} \; , \quad i = 1, \cdots, N_\alpha ,$$

with the "initial conditions"

$$A_{i-1(0)}^{\beta\alpha} = -V_{N_\beta i}^{\beta\alpha} \; , \quad i = 1, \cdots, N_\alpha \; ,$$

following from a comparison of (5.31) with (5.32). It is easy to see, that these recursion relations determine the complete solution, from which the Lagrangian gauge symmetries can be obtained. The result is summarized in the following table, where the entries are the coefficients $A_{i(n)}^{\beta\alpha}$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$\cdots$
$i = N_\alpha$	$\delta^{\beta\alpha}$	0	0	0	$\cdots$	$\cdots$
$i = N_\alpha - 1$	$-V_{N_\beta, N_\alpha}^{\beta\alpha}$	$\delta^{\beta\alpha}$	0	$\cdots$	$\cdots$	$\cdots$
$i = N_\alpha - 2$	$-V_{N_\beta, N_\alpha - 1}^{\beta\alpha}$	$-V_{N_\beta N_\alpha}^{\beta\alpha}$	$\delta^{\beta\alpha}$	0	$\cdots$	$\cdots$
$i = N_\alpha - 3$	$-V_{N_\beta, N_\alpha - 2}^{\beta\alpha}$	$-V_{N_\beta N_\alpha - 1}^{\beta\alpha}$	$-V_{N_\beta N_\alpha}^{\beta\alpha}$	$\delta^{\beta\alpha}$	0	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

With ϵ^A and Ω_A replaced by $\epsilon_i^{(\alpha)}$ and $\phi_i^{(\alpha)}$ respectively, the infinitesimal gauge transformations $\delta q^\ell = \sum_{\alpha,i} \epsilon_i^{(\alpha)} \{q^\ell, \phi_i^{(\alpha)}\}$ take the form

$$\delta q^\ell = \sum_{\beta=1}^{M} \sum_{n \geq 0} \frac{d^n \alpha^\beta}{dt^n} \rho_{(n)\beta}^\ell (q, p) \; ,$$

with

$$\rho_{(n)\beta}^\ell (q, p) = \sum_{\alpha=1}^{M} \sum_{j \geq 0} \theta(N_\alpha - n - j) A_{j(n)}^{\beta\alpha} \frac{\partial \phi_j^{(\alpha)}}{\partial p_\ell} \; ,$$

where θ is the Heaviside theta-function with $\theta(0) = 1$, and where it is understood that the dependence on the canonical momenta on the rhs has been replaced by the respective expressions in terms of the Lagrangian variables.

In the case where all the families contain at most two members, and where the structure functions $U_{aB}^{\ \gamma}$ vanish [6] we can relax the above assumption concerning the constancy of the structure functions $V_{ij}^{\alpha\beta}$, since our iterative scheme already terminates with equation (5.31) with $N_\alpha = 1$ for all α. With (5.4) an infinitesimal gauge transformation δq^i then takes the form

$$\delta q^\ell = \sum_\beta \alpha^\beta \{q^\ell, \phi_0^{(\beta)}\} + \sum_{\beta=1}^{M} \left(\frac{d\alpha^\beta}{dt} - \sum_{\gamma=1}^{M} \alpha^\gamma V_{11}^{\gamma\beta} \right) \{q^\ell, \phi_1^{(\beta)}\} \ .$$

Example

The following modified version of the Nambu-Goto model [Hwang 1983] has the above mentioned features.

Consider the action

$$S = \int d\sigma \left(\frac{1}{2}\frac{\dot{x}^2}{\lambda} - \frac{\mu}{\lambda}\dot{x}x' + \frac{1}{2}\frac{\mu^2}{\lambda}x'^2 - \frac{1}{2}\lambda x'^2 \right) \ , \tag{5.34}$$

where the 4-vector $x^\mu(\tau,\sigma)$ labels the coordinates of a "string" parametrized by τ and σ, with the "dot" and "prime" denoting the derivative with respect to τ and σ, respectively. There are two primary constraints, $\pi_1 = 0$ and $\pi_2 = 0$, where π_1 and π_2 are the momenta conjugate to the fields $\lambda(\tau,\sigma)$ and $\mu(\tau,\sigma)$, respectively. Hence in our notation

$$\phi_0^{(1)} = \pi_1 \ , \quad \phi_0^{(2)} = \pi_2 \ .$$

The canonical Hamiltonian on the primary surface reads,

$$H_0 = \int d\sigma \{ \frac{\lambda}{2}(p^2 + x'^2) + \mu p \cdot x' \} \ , \tag{5.35}$$

where p_μ is the four-momentum conjugate to the coordinate x^μ. The conservation in time of the primary constraints leads respectively to secondary constraints, which in our notation read

$$\phi_1^{(1)} = \frac{1}{2}(p^2 + x'^2) = 0 \ , \quad \phi_1^{(2)} = p \cdot x' = 0 \ .$$

[6] Examples are the $U(1)$ Maxwell theory, the $SU(N)$ Yang-Mills theories, as well as the example discussed next.

One readily checks that there are no further constraints. These secondary constraints are just the primary constraints of the Nambu-Goto string model. They satisfy the Poisson brackets. [7] (we suppress the τ variable)

$$
\begin{aligned}
\{\phi_1^{(1)}(\sigma),\phi_1^{(1)}(\sigma')\} &= \phi_1^{(2)}(\sigma)\partial_\sigma\delta(\sigma-\sigma') - \phi_1^{(2)}(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma')\ , \\
\{\phi_1^{(1)}(\sigma),\phi_1^{(2)}(\sigma')\} &= \phi_1^{(1)}(\sigma)\partial_\sigma\delta(\sigma-\sigma') - \phi_1^{(1)}(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma') \\
\{\phi_1^{(2)}(\sigma),\phi_1^{(2)}(\sigma')\} &= \phi_1^{(2)}(\sigma)\partial_\sigma\delta(\sigma-\sigma') - \phi_1^{(2)}(\sigma')\partial_{\sigma'}\delta(\sigma-\sigma')\ .
\end{aligned}
\tag{5.36}
$$

All other Poisson brackets vanish. The constraints are seen to be first class. In our terminology, we thus have two families, each with two members.

Note that the Hamiltonian (5.35) is of the form

$$
H_0 = \int d\sigma\left(\lambda\phi_1^{(1)}(\sigma) + \mu\phi_1^{(2)}(\sigma)\right)\ , \tag{5.37}
$$

i.e., it is just a linear combination of the level one constraints. It is therefore called a "zero-Hamiltonian". The structure functions $V_{ij}^{\alpha\beta}$, defined in (5.27), are read off from the Poisson brackets

$$
\{H_0,\phi_0^{(1)}\} = \phi_1^{(1)}\ , \qquad [H_0,\phi_0^{(2)}] = \phi_1^{(2)}\ ,
$$

and

$$
\begin{aligned}
\{H_0,\phi_1^{(1)}\} &= -\lambda\partial_\sigma\phi_1^{(2)} - 2\lambda'\phi_1^{(2)} - \mu\partial_\sigma\phi_1^{(1)} - 2\mu'\phi_1^{(1)}\ , \\
\{H_0,\phi_1^{(2)}\} &= -\lambda\partial_\sigma\phi_1^{(1)} - 2\lambda'\phi_1^{(1)} - \mu\partial_\sigma\phi_1^{(2)} - 2\mu'\phi_1^{(2)}\ .
\end{aligned}
$$

They are given by

$$
\begin{aligned}
V_{11}^{11}(\sigma,\sigma') = V_{11}^{22}(\sigma,\sigma') &= -\left(\mu(\sigma)\partial_\sigma + 2\mu'(\sigma)\right)\delta(\sigma-\sigma')\ , \\
V_{11}^{12}(\sigma,\sigma') = V_{11}^{21}(\sigma,\sigma') &= -\left(\lambda(\sigma)\partial_\sigma + 2\lambda'(\sigma)\right)\delta(\sigma-\sigma')\ .
\end{aligned}
$$

Since for the example in question the maximum levels for the two chains generated from $\phi_0^{(1)}$ and $\phi_0^{(2)}$ are $N_1 = N_2 = 1$, it follows from (5.31) that our iterative scheme for finding the solution to (5.28) already ends at the first step, with $\epsilon_0^{(\alpha)}$ given by

$$
\epsilon_0^{(\alpha)} = \frac{d\alpha^\alpha}{d\tau} - \int d\sigma' \sum_{\beta=1}^{2} \alpha^\beta(\sigma') V_{11}^{\beta\alpha}(\sigma',\sigma)\ .
$$

[7] In obtaining these Poisson bracket relations, identities such as

$$
f(\sigma')\partial_\sigma\delta(\sigma-\sigma') = (\partial_\sigma f(\sigma))\delta(\sigma-\sigma') + f(\sigma)\partial_\sigma\delta(\sigma-\sigma')
$$

have been used.

We thus obtain

$$\epsilon_0^{(1)} = \frac{d\alpha^1}{d\tau} - \mu\partial_\sigma\alpha^1 + \mu'\alpha^1 - \lambda\partial_\sigma\alpha^2 + \lambda'\alpha^2 \ ,$$
$$\epsilon_0^{(2)} = \frac{d\alpha^2}{d\tau} - \mu\partial_\sigma\alpha^2 + \mu'\alpha^2 - \lambda\partial_\sigma\alpha^1 + \lambda'\alpha^1 \ . \tag{5.38}$$

From here and (5.4) we now compute the corresponding transformation laws for the fields to be

$$\delta x^\mu = \alpha^1 p^\mu + \alpha^2\partial_\sigma x^\mu \ , \tag{5.39}$$
$$\delta\lambda = \epsilon_0^{(1)} \ , \quad \delta\mu = \epsilon_0^{(2)} \ . \tag{5.40}$$

Making use of the expressions for $\epsilon_0^{(\alpha)}$ derived above, we verify that these transformation laws agree with that quoted in the literature [Batlle 1990].

5.7 Reparametrization invariant approach

In the previous section we considered a restricted class of gauge transformations (5.4), with a parametrization tied to first class constraints generated recursively according to (5.25). In this section we consider transformations which are form invariant under redefinitions of the constraints. As before, we shall concentrate on purely first class systems.

Consider the *total* action

$$S_T = \int_{t_1}^{t_2} dt \ [p_i\dot{q}^i - H(q,p) - v^\alpha\phi_\alpha] \ . \tag{5.41}$$

Next consider a variation δS_T. Making use of $\delta\dot{q}^i = \frac{d}{dt}\delta q^i$ one obtains, after dropping a total time derivative

$$\delta S_T = \int dt \left[\left(\dot{q}^i - \frac{\partial H}{\partial p_i} - v^\alpha\frac{\partial\phi_\alpha}{\partial p_i}\right)\delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q^i} + v^\alpha\frac{\partial\phi_\alpha}{\partial q^i}\right)\delta q^i - \delta v^\alpha\phi_\alpha\right] .$$

The Hamilton equations of motion, together with the primary constraints follow by requiring that δS_T vanishes for arbitrary variations δq^i, δp_i and δv^α. For a local symmetry transformation, $q^i \to q^i + \delta q^i$, $p_i \to p_i + \delta p_i$ this variation must vanish *without* making use of the equations of motion. Consider a variation of the form

$$\delta q^i = \frac{\partial G}{\partial p_i} = \{q^i, G\} \, , \quad \delta p_i = -\frac{\partial G}{\partial q^i} = \{p_i, G\} \, , \tag{5.42}$$

with

$$G(q,p,t) = \epsilon^A(q,p,t)\Omega_A(q,p) \ . \tag{5.43}$$

On the constrained surface the transformations (5.42) evidently coincide with the corresponding variations (5.4). One readily finds that

$$\delta S_T = \int dt \left(\frac{\partial G}{\partial t} - \{H, G\} - v^\alpha \{\phi_\alpha, G\} - \delta v^\alpha \phi_\alpha \right) ,$$

where we have dropped a total derivative $\frac{dG}{dt}$. The total action is trivially invariant if [8]

$$\frac{\partial G}{\partial t} - \{H, G\} - v^\alpha \{\phi_\alpha, G\} - \delta v^\alpha \phi_\alpha = 0 . \tag{5.44}$$

Note that the choice of the first class constraints in (5.43) is left unspecified, and in particular depends on the algorithm generating them. Thus any linear combination of the constraints, with non-singular q and p dependent coefficients is still first class. Any redefinition of the constraints can however be absorbed into a redefinition of the functions ϵ^A. Hence the equations determining the ϵ^A's must be form invariant under reparametrization. For this it is important, that the q and p dependence of the ϵ^A's be taken into account when calculating the Poisson brackets.

Consider now equation (5.44). As always the primary and secondary constraints are labeled by small Greek and Latin indices, respectively. Since, by assumption, all the constraints are first class, eq. (5.44) implies the following equations for the parameters $\epsilon^A = \epsilon^A(q, p, t)$:

$$\frac{\partial \epsilon^a}{\partial t} - \epsilon^B \{V_B^a + v^\beta(t) U_{\beta B}^a\} + \{\epsilon^a, H\} + v^\beta(t)\{\epsilon^a, \phi_\beta\} \approx 0 , \tag{5.45}$$

$$\delta v^\alpha \approx \frac{\partial \epsilon^\alpha}{\partial t} - \epsilon^B \{V_B{}^\alpha + v^\beta(t) U_{\beta B}^\alpha\} + \{\epsilon^\alpha, H\} + v^\beta(t)\{\epsilon^\alpha, \phi_\beta\} , \tag{5.46}$$

which are to be compared with the on-shell relations (5.14) and (5.15). Thus equations (5.45) and (5.46) have the same form as (5.14) and (5.15) on the space of solutions, since in this case

$$\frac{\partial \epsilon^A}{\partial t} + \{\epsilon^A, H\} + v^\alpha \{\epsilon^A, \phi_\alpha\} \approx \frac{d\epsilon^A}{dt} .$$

One readily verifies that equations (5.45) and (5.46) are form invariant under the transformations

$$\Omega_A(q, p) \to \tilde{\Omega}_A(q, p) = \Lambda_A{}^B(q, p) \Omega_B(q, p) ,$$

[8]This equation can also be written in the compact form [Banerjee 2000b],

$$\frac{\partial G}{\partial t} - \{H_T, G\} = 0 ,$$

if one agrees to *formally* define $\delta v^\alpha \equiv \{v^\alpha, G\}$.

$$\epsilon^A(q,p) \to \tilde{\epsilon}^A(q,p) = \epsilon^B(q,p)(\Lambda^{-1})_B^{\ A}(q,p) \ ,$$
$$v^\alpha \to \tilde{v}^\alpha = v^\beta(\Lambda^{-1})_\beta^{\ \alpha}(q,p) \ ,$$

and

$$V_B^{\ A}(q,p) \to \tilde{V}_B^{\ A}(q,p) \, , \quad U_{\beta B}^A(q,p) \to \tilde{U}_{\beta B}^A(q,p) \ ,$$

where $\tilde{U}_{\beta B}^A(q,p)$ and $\tilde{V}_B^{\ A}(q,p)$ are defined in terms of the constraints $\tilde{\Omega}_A$ by equations analogous to (5.12) and (5.13).

We now look for solutions to (5.45). In the following we will restrict ourselves to systems with only *one* primary constraint $\Omega_1 = 0$. If the (first class) secondary constraints are generated successively by the Dirac algorithm according to (5.25), starting with the primary constraint, i.e.

$$\Omega_{A+1} = \{H, \Omega_A\} \ \ ; \ A \leq M-1 \, , \tag{5.47}$$

with M the total number of (first class) constraints, then [9]

$$U_{1B}^A = 0 \quad \text{for } A > B \, .$$

Note that A and B are further restricted by $A \leq M$, $B \leq M$, and that the non-vanishing components of $V_B^{\ A}$ are given by

$$V_{A-1}^{\ A} = 1; \quad A = 2, \cdots, M$$

and possibly $V_M^{\ A}$, $A = 1, \cdots, M$. Making use of this information, one derives the following recursion relation

$$\begin{aligned} \epsilon^{M-n} = \left(W_{M-n}^{M-n+1}\right)^{-1} \Bigg[& \frac{\partial \epsilon^{M-n+1}}{\partial t} + \{\epsilon^{M-n+1}, H\} + v\{\epsilon^{M-n+1}, \Omega_1\} \\ & - \sum_{A=M-n+1}^{M} \epsilon^A W_A^{M-n+1} \Bigg] \ ; \ n = 1, \ldots, M-1 \ , \end{aligned} \tag{5.48}$$

where

$$W_B^A(q,p) = V_B^{\ A}(q,p) + vU_{1B}^A(q,p) \ .$$

Equations (5.48) cannot be solved in general. Solutions can however be obtained for the case where ϵ^M is only an explicit function of t: $\epsilon^M = \alpha(t)$. Then the above equation allows one to determine ϵ^{M-1} as a function of $v(t)$, $\alpha(t)$, $\dot{\alpha}(t)$, and the canonical variables, since in this case the second and third terms on the rhs of (5.48) vanish for $n = 1$. Thus ϵ^{M-1} is now determined. This

[9] Persistence of the constraints in time requires that $\{\Omega_A, H_T\} \approx 0$. Hence, following the Dirac algorithm (5.25), a new constraint is generated from Ω_B only if $\{\Omega_1, \Omega_B\} = \sum_A U_{1B}^A \Omega_A$ vanishes on the subspace defined by $\Omega_1 = 0$, $\Omega_2 = 0$, $\cdots, \Omega_B = 0$, i.e., if $U_{1B}^A = 0$ for $A > B$.

allows one to descend to ϵ^{M-2}. Proceeding in this way we find that ϵ^{M-n}, $n = 1, \cdots, M-1$ takes the form

$$\epsilon^{M-n} = \sum_{\ell=0}^{n} \rho_\ell^{(M-n)}(q, p; v, \dot{v}, \cdots, d^{n-1}v/dt^{n-1}) \frac{d^\ell \alpha}{dt^\ell} \ .$$

Correspondingly the transformation law for the coordinates and momenta have the form

$$\delta q^i = \sum_{k=1}^{M} \sum_{\ell=0}^{M-k} f_{i\ell}^{(k)}(q, p; v, \dot{v}, \cdots, d^{M-k-1}v/dt^{M-k-1}) \frac{d^\ell \alpha}{dt^\ell} \ , \tag{5.49}$$

$$\delta p_i = \sum_{k=1}^{M} \sum_{\ell=0}^{M-k} h_{i\ell}^{(k)}(q, p; v, \dot{v}, \cdots, d^{M-k-1}v/dt^{M-k-1}) \frac{d^\ell \alpha}{dt^\ell} \ . \tag{5.50}$$

The above variations ensure that the total action associated with an arbitrary phase space trajectory $q(t), p(t)$ remains invariant. Thus if $q^i(t)$, $p_i(t)$ minimize the total action, then this is also true for $q^i + \delta q^i$, $p_i + \delta p_i$, and v replaced by $v + \delta v$, for a suitable choice of δv.

We remark that the (off shell) transformations (5.49) and (5.50) do not leave the total Hamiltonian invariant for functions $\epsilon^A(q, p, t)$ which depend explicitly on time. This follows immediately from

$$\delta H_T = \{H, G\} + v^\alpha \{\phi_\alpha, G\} + \delta v^\alpha \phi_\alpha$$

which for ϵ^A and δv^α satisfying (5.45) and (5.46), and hence (5.44), implies that (see (5.44))

$$\delta H_T = \sum_A \frac{\partial \epsilon^A}{\partial t} \Omega_A \ ,$$

which only vanishes weakly.

The solutions to the fundamental equation (5.45) have been obtained assuming that ϵ^M is only a function of time. The corresponding transformation laws for the coordinates and momenta are then functions of the multiplier $v(t)$ and time derivatives thereof, the canonical variables, as well as of explicit time.

Finally let us observe that both transformation laws discussed above, i.e., (5.4) and (5.42), are consistent with the commutativity $\delta \dot{q}^i(t) = \frac{d}{dt}\delta q^i$ on the level of the solutions to the equations of motion. Thus consider the difference

$$\Delta = \left(\frac{d}{dt}\delta q^i - \delta \dot{q}^i \right) \tag{5.51}$$

with

$$\delta q^i = \{q^i, G\} = \epsilon^A \{q^i, \Omega_A\} + \{q^i, \epsilon^A\} \Omega_A \ .$$

Hence, using the Hamilton equations of motion,

$$\frac{d}{dt}\delta q^i \approx \frac{d\epsilon^A}{dt}\{q^i,\Omega_A\} + \epsilon^A\{\{q^i,\Omega_A\},H\} + v^\alpha\epsilon^A\{\{q^i,\Omega_A\},\phi_\alpha\}\ ,$$

where we have made use of the persistency equation $\frac{d\Omega_A}{dt} \approx 0$. Furthermore, since ϵ^A are in general functions of q, p, t we have

$$\frac{d\epsilon^A}{dt} \approx \frac{\partial\epsilon^A}{\partial t} + \{\epsilon^A, H_T\}\,.$$

Making use of the Jacobi identity (3.23) we have that

$$\frac{d}{dt}\delta q^i \approx \left[\frac{\partial\epsilon^A}{\partial t} - \{H_T,\epsilon^A\}\right]\{q^i,\Omega_A\} + \epsilon^A\{\{H_T,\Omega_A\},q_i\} + \epsilon^A\{\{q_i,H_T\},\Omega_A\}\ ,$$

where the Lagrange multipliers in H_T do not participate in the computation of the Poisson brackets.

On the other hand, again making use of the equations of motion we have that

$$\delta\dot{q}^i \approx \{\{q^i,H_T\},G\} + \delta v^\alpha\{q^i,\phi_\alpha\}\ ,$$

so that on Γ, making use of (5.12) and (5.13), (5.51) is given by

$$\Delta \approx \left[\frac{\partial\epsilon^A}{\partial t} - \{H_T,\epsilon^A\} - \epsilon^B(V^A_B + v^\beta U^A_{\beta B}) - \delta v^\alpha\delta_{\alpha A}\right]\{q^i,\Omega_A\}\ ,$$

where $\Omega_\alpha \equiv \phi_\alpha$. This expression vanishes on account of (5.45) and (5.46). A similar expression with q^i replaced by p_i is obtained for $\left(\frac{d}{dt}\delta p_i - \delta\dot{p}_i\right)$.

Example

We now give an example which demonstrates how a reparametrization invariant formulation can be used to simplify the transformation laws. Consider the Lagrangian (2.6), i.e.,

$$L = \frac{1}{2}(\dot{q}_2 - e^{q_1})^2 + \frac{1}{2}(\dot{q}_3 - q_2)^2\,.$$

There exists just one primary constraint $\Omega_1 := p_1 = 0\,.$ The total Hamiltonian is thus given by $H_T = H + vp_1$, where

$$H = \frac{1}{2}p_2^2 + \frac{1}{2}p_3^2 + e^{q_1}p_2 + q_2p_3\,.$$

Persistence in time of the primary constraint leads to the secondary constraint

$$\Omega_2 := \{H, \Omega_1\} = e^{q_1} p_2 = 0\,,$$

which in turn leads to a tertiary constraint

$$\Omega_3 = \{H, \Omega_2\} = e^{q_1} p_3 = 0\,. \tag{5.52}$$

There are no further constraints. All the constraints are first class. Note that we have defined the constraints in accordance with the scheme (5.47). We are free to do so, since any other realization of the constraints is equivalent to redefining the parameters in the definition of the generator (5.43). [10] The only non-vanishing structure functions $V_B{}^A$ and U_{1B}^A are given by

$$\begin{aligned} V_1{}^2 &= V_2{}^3 = 1\ , \\ U_{12}^2 &= U_{13}^3 = -1\ . \end{aligned}$$

A solution to (5.48) can now be obtained with the Ansatz, that the parameter ϵ^3, associated with the last constraint generated from the primary, is only a function of time, i.e., $\epsilon^3 = \alpha(t)$. At this point we have clearly made a special choice tied to a particular realization of the constraints. Having obtained a solution, we are then free to realize the constraints in different ways, thereby redefining the parameters. With the above choice of constraints, and setting $\epsilon^3 = \alpha(t)$ one finds for the solution of (5.48)

$$\begin{aligned} \epsilon^3 &= \alpha\ , \\ \epsilon^2 &= \dot\alpha + v\alpha, \\ \epsilon^1 &= \ddot\alpha + 2v\dot\alpha + \dot v\alpha + \alpha v^2\ , \\ \delta v &= \frac{d^3\alpha}{dt^3} + 2v\ddot\alpha + 3\dot\alpha\dot v + v^2\dot\alpha + 2\alpha v\dot v + \alpha\ddot v = \dot\epsilon^1\ . \end{aligned} \tag{5.53}$$

Note that the parameters do not depend on the canonical variables, but only on $\alpha(t)$, the Lagrange multiplier v, and time derivatives thereof. It then follows from $\dot q^1 \approx \{q^1, H_T\}$ that $v = \dot q^1$. Hence, $\delta v = \delta \dot q_1$. On the other hand $\delta q_1 = \{q_1, G\} = \epsilon^1$, where G is the generator $G = \epsilon^A \Omega_A$. Commutativity of the time derivative and δ-operations implies that $\delta\dot q_1 = \delta v = \dot\epsilon^1$, which coincides with the rhs of (5.53).

By redefining the constraints, the identification $v = \dot q_1$ may be lost, and the parameters ϵ^B may be replaced by expressions depending on the canonical variables. Thus consider e.g. the following reparametrization of the constraints

$$\Omega_1 \to \bar\Omega_1 = e^{-q_1} p_1\ ,$$

[10] In example 3 of section 3.2.3 we had chosen the constraints to be $p_1 = p_2 = p_3 = 0$.

$$
\begin{aligned}
\Omega_2 &\to \bar{\Omega}_2 = p_2 \ , \\
\Omega_3 &\to \bar{\Omega}_3 = p_3 \ ,
\end{aligned}
$$

which, in particular, implies that $v \to \bar{v} = e^{q_1} v = e^{q_1} \dot{q}_1$. The generator is now given by $G = \bar{\epsilon}^A \bar{\Omega}_A$, where the new parameters $\bar{\epsilon}^A$ now take the form: $\bar{\epsilon}^3 = e^{q_1} \alpha$, $\bar{\epsilon}^2 = e^{q_1} \epsilon^2$ $\bar{\epsilon}^1 = e^{q_1} \epsilon^1$. More explicitly one has that

$$
\begin{aligned}
\bar{\epsilon}^3 &= e^{q_1} \alpha \\
\bar{\epsilon}^2 &= \frac{d}{dt} \left(e^{q_1} \alpha \right) \\
\bar{\epsilon}^1 &= \frac{d^2}{dt^2} \left(e^{q_1} \alpha \right) \\
\delta \bar{v} &= \frac{d^3}{dt^3} \left(e^{q_1} \alpha \right) \ .
\end{aligned}
$$

The fact, that the expressions (5.45) and (5.46) also involve time derivatives suggests [Costa 1985] that the observables $\mathcal{O}(q,p)$ of a theory described by a singular Lagrangian, should possess the property (5.3), although G only generates symmetry transformations provided that the gauge parameters satisfy the relations (5.46). One refers to this as the "Dirac conjecture". In the following chapter we will discuss in detail in which sense this conjecture holds.

We close this chapter with a remark. The generalization of the above symmetry considerations to systems with first and second class constraints is straightforward. For mixed systems we had shown in chapter 3 that the Hamilton equations of motion for an arbitrary function of the canonical variables can be written in the form (3.69). The gauge degrees of freedom are contained entirely in the sum over α_1 on the rhs. These equations of motion are equivalent to

$$
\dot{f} \approx \{f, \bar{H}_T\}
$$

where

$$
\bar{H}_T = H^{(1)} + v^{\alpha_1} \phi^{(1)}_{\alpha_1}
$$

with $H^{(1)}$ defined in (3.65). One thus only has to replace H by $H^{(1)}$ in all our expressions above, in order to include also the second class constraints. This means of course that the coefficient functions ${V_A}^B$ must also be computed from (5.13) with H replaced by $H^{(1)}$.

Chapter 6

The Dirac Conjecture

6.1 Introduction

In this chapter we shall make our earlier statements concerning "Dirac's conjecture" more precise. This conjecture states that the generators of the local symmetries of the action are given by the complete set of first class constraints. In the following we will restrict ourselves to purely first class systems. [1] Hence $H^{(1)} = H$ in (3.65). Starting from an equivalent first order Lagrangian formulation, where the Euler-Lagrange equations are just the Hamilton equations of motion, we make use of the Lagrangian methods discussed in chapter 2, in order to establish a direct connection between the gauge identities generated by the Lagrangian algorithm and the generators of gauge symmetries of the total action. [2] We then discuss several examples, which have been cited in the literature as counterexamples to Dirac's conjecture, in order to point out some subtleties, and show that the Dirac conjecture holds, if properly interpreted.

6.2 Gauge identities and Dirac's conjecture

Before embarking on a detailed discussion of the Dirac conjecture, we want to provide the reader with a guide of our line of reasoning.

As mentioned above, our aim is to use purely Lagrangian methods to obtain the transformation laws for the coordinates, momenta and Lagrange multipliers which leave the *total* phase-space action invariant. As we will see, these transformation laws are of the form (5.4), where i) all the first class constraints

[1] The case of a mixed system has been studied in [Rothe 2003c].

[2] For an alternative discussion of Dirac's conjecture see [Castellani 1982/90, Di Stefano 1983, Costa 1985, Cabo 1986, Gracia 1988, Gitman 1990].

must be included on the rhs, and ii) the constraints are generated in a particular way. To show this we will first construct a Lagrangian $L(Q,\dot{Q})$, linear in the velocities $\dot{Q}$, so that its Euler derivatives lead to the Hamilton equations of motion of the theory in question, if the coordinates Q_a are properly identified with the phase-space variables associated with the second order Lagrangian of primary interest. We can then study the symmetries of the corresponding action making use of the Lagrangian methods developed in chapter 2. Proceeding as in that chapter, one searches for left zero modes of the kinetic term of the various Euler derivatives generated iteratively starting from the zeroth level. Crucial for the analysis is that at each level the Euler derivatives and zero modes have a generic structure determined by the first class constraints of the theory. The algorithm stops once no new zero modes are generated, and one is left with only gauge identities. From these gauge identities one then extracts the transformation laws for the Q_a's, and correspondingly for the coordinates q_i and momenta p_i of the Hamiltonian formulation. These take the form (5.4) with the ϵ^A's restricted by (5.14). While the form (5.4) of the transformation laws will be strictly proven, the last mentioned property will only be demonstrated explicitly for an arbitrary system with two primary, and one secondary constraint. That it must be true in general follows from the fact that, by construction, the transformations leave the total action invariant. It is nevertheless instructive to demonstrate this for a particular type of system.

We now prove the above made assertions [Rothe 2002, 2003c, 2004]. [3] Since our analysis proceeds along the lines of chapter 2, we will denote, as in that chapter, the level ℓ of the algorithm by a corresponding superscript. The labels of the coordinates will however differ from those in chapter 2 for reasons which will become clear.

Let $\phi_\alpha^{(0)} = 0$ $(\alpha = 1, 2, \cdots, N_0)$ be the primary constraints associated with a *second* order Lagrangian $L(q,\dot{q})$, [4] where $q_i, (i = 1, \cdots, n)$ are coordinates in configuration space, and let $H_0(q,p)$ be the corresponding canonical Hamiltonian defined on the primary surface. One readily verifies that the Euler-Lagrange equations associated with the first order (total) Lagrangian

$$L_T(q,p,\dot{q},\dot{p},\lambda,\dot{\lambda}) = \sum_{i=1}^{n} p_i\dot{q}_i - H_T(q,p,\lambda) \tag{6.1}$$

[3]Reference [Rothe 2002] contains some printing errors. In particular the index "a" in Eq. (56) of that reference takes the values $a = 2, \cdots, M$, and α in Eq. (59) runs over $1, 2, \cdots, 2n+1$, and not from 1 to 7, as stated in the paragraph following (59).

[4]Notice that the labeling of the primary and secondary constraints differs from that in chapter 5, and follows that of [Rothe 2004], except that latin indices are replaced by greek indices, in agreement with the conventions followed in this book.

with

$$H_T(q,p,\lambda) = H_0(q,p) + \sum_{\alpha=1}^{N_0} \lambda_\alpha \phi_\alpha^{(0)} \,,$$

reproduces the Hamilton equations of motion including the primary constraints, if we regard q_i, p_i and λ_α as coordinates in a $2n+N_0$ dimensional configuration space. We now write (6.1) in the form

$$L_T = \sum_{a=1}^{2n+N_0} a_a(Q)\dot{Q}_a - H_T(Q)\,, \tag{6.2}$$

where

$$Q_a := (\vec{q}, \vec{p}, \lambda_1, \cdots, \lambda_{N_0})$$

and

$$H_T(Q) = H(Q_1, \cdots, Q_{2n}) + \sum_{\alpha=1}^{N_0} Q_{2n+\alpha}\phi_\alpha^{(0)} \,.$$

The non-vanishing elements of a_a are given by $a_i = Q_{n+i} = p_i$ $(i = 1, \cdots, n)$. The $2n + N_0$ components of the Euler derivative are

$$\begin{aligned} E_a^{(0)} &= \frac{d}{dt}\left(\frac{\partial L_T}{\partial \dot{Q}_a}\right) - \frac{\partial L_T}{\partial Q_a} \\ &= -\sum_{b=1}^{2n+N_0} W_{ab}^{(0)} \dot{Q}_b + K_a^{(0)} \,, \end{aligned} \tag{6.3}$$

with

$$W_{ab}^{(0)} = \partial_a a_b - \partial_b a_a \ ,$$

the $(2n + N_0) \times (2n + N_0)$ matrix

$$\mathbf{W^{(0)}} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} & \vec{0} \cdots \vec{0} \\ \mathbf{1} & \mathbf{0} & \vec{0} \cdots \vec{0} \\ \vec{0}^T & \vec{0}^T & 0 \cdots 0 \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ \vec{0}^T & \vec{0}^T & 0 \cdots 0 \end{pmatrix} ,$$

and

$$K_a^{(0)} = \frac{\partial H_T}{\partial Q_a} \,,$$

where $\mathbf{1}$ is an $n \times n$ unit matrix, $\vec{0}$ are N_0-component null column vectors (associated with the absence of $\dot{\lambda}_\alpha$ in L_T), and $\vec{0}^T$ is the transpose of $\vec{0}$.

The variation of the total action

$$S_T = \int dt\ L_T(Q, \dot{Q}) \tag{6.4}$$

is given by

$$\delta S_T = -\sum_a \int dt\ E_a^{(0)} \delta Q_a\,, \tag{6.5}$$

where we have dropped a boundary term. The left-zero modes of $\mathbf{W}^{(0)}$ are given by

$$\vec{v}^{(0)}(\alpha) = \left(\vec{0}, \vec{0}, \hat{n}(\alpha)\right)\ ,$$

where $\hat{n}(\alpha)$ is an N_0-component unit vector with the only non-vanishing component in the αth place. Hence

$$\vec{v}^{(0)}(\alpha) \cdot \vec{E}^{(0)} = \vec{v}^{(0)}(\alpha) \cdot \vec{K}^{(0)} = \phi_\alpha^{(0)}\,. \tag{6.6}$$

We thus recover on-shell the (first class) primary constraints $\phi_\alpha^{(0)} = 0$.

We now adjoin the time derivative of the primaries $\{\phi_\alpha^{(0)}\}$ to $\vec{E}^{(0)}$ and construct the $2n + 2N_0$ component (level one) vector $\vec{E}^{(1)}$:

$$\vec{E}^{(1)} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\vec{\phi}^{(0)} \end{pmatrix}\,, \tag{6.7}$$

where $\vec{\phi}^{(0)}$ is a column vector with components $(\phi_1^{(0)}, \cdots, \phi_{N_0}^{(0)})$. $\vec{E}^{(1)}$ vanishes on shell, i.e. for $\vec{E}^{(0)} = 0$. The components of $\vec{E}^{(1)}$, which we label by a_1, can be written in the form

$$E_{a_1}^{(1)} = -\sum_a W_{a_1 a}^{(1)} \dot{Q}_a + K_{a_1}^{(1)}(Q)\,,$$

where

$$\vec{K}^{(1)} = \begin{pmatrix} \vec{K}^{(0)} \\ \vec{0} \end{pmatrix}\,,$$

and $\mathbf{W^{(1)}}$ is now the *rectangular* matrix

$$\mathbf{W}^{(1)} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} & \vec{0}\cdots\vec{0} \\ \mathbf{1} & \mathbf{0} & \vec{0}\cdots\vec{0} \\ \vec{0}^T & \vec{0}^T & 0\cdots 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \vec{0}^T & \vec{0}^T & 0\cdots 0 \\ -\nabla\phi_1^{(0)} & -\tilde{\nabla}\phi_1^{(0)} & 0\cdots 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ -\nabla\phi_{N_0}^{(0)} & -\tilde{\nabla}\phi_{N_0}^{(0)} & 0\cdots 0 \end{pmatrix} .$$

Here

$$\nabla := (\partial_1, \cdots, \partial_n)\,, \quad \tilde{\nabla} := (\partial_{n+1}, \cdots, \partial_{2n})\,.$$

We seek new constraints by looking for left zero modes of $\mathbf{W^{(1)}}$. They are N_0 in number, and are given by

$$\vec{v}^{(1)}(\alpha) := \left(-\tilde{\nabla}\phi_\alpha^{(0)}, \nabla\phi_\alpha^{(0)}, \vec{0}, \hat{e}^{(0)}(\alpha)\right)\,, \tag{6.8}$$

where $\vec{0}$ is an N_0-component null vector, and $\hat{e}^{(0)}(\alpha)$ is an N_0-component unit vector, with the only non-vanishing component in the αth position. This leads to

$$\begin{aligned} \vec{v}^{(1)}(\alpha)\cdot\vec{E}^{(1)} = \vec{v}^{(1)}(\alpha)\cdot\vec{K}^{(1)} &= \sum_i \left(\frac{\partial\phi_\alpha^{(0)}}{\partial q_i}\frac{\partial H_T}{\partial p_i} - \frac{\partial H_T}{\partial q_i}\frac{\partial\phi_\alpha^{(0)}}{\partial p_i}\right) \\ &= \{\phi_\alpha^{(0)}, H_T\}\,, \end{aligned}$$

or

$$\vec{v}^{(1)}(\alpha)\cdot\vec{E}^{(1)} = \{\phi_\alpha^{(0)}, H\} - \sum_\gamma \lambda_\gamma\{\phi_\gamma^{(0)}, \phi_\alpha^{(0)}\}\,, \tag{6.9}$$

which by constuction vanish on shell (i.e., for $\vec{E}^{(0)} = \vec{0}$). Poisson brackets will always be understood to be taken with respect to the canonically conjugate variables q_i and p_i. In order to determine the constraints and gauge identities we can proceed iteratively either "chain by chain" [Loran 2002], or "level by level". For the purpose of illustration let us adopt the former procedure. Since we are studying a purely first class system, (6.9) will imply a new constraint, provided that $\{\phi_\gamma^{(0)}, \phi_\alpha^{(0)}\}$ is a linear combination of primaries. [5] In that case we have a new constraint which we define in the *strong* sense:

$$\phi_\alpha^{(1)} := \{\phi_\alpha^{(0)}, H\} \ . \tag{6.10}$$

Notice that in the chain by chain procedure of generating the constraints the subscript α labels the parent of the chain (i.e., a primary constraint) and thus always takes the values $1, \cdots, N_0$, whereas the upper index labels the "decendents" whose number L_α is determined by the parent. From (6.9) and (6.10) we have that,

$$\phi_\alpha^{(1)} = \vec{v}^{(1)}(\alpha) \cdot \vec{E}^{(1)} + \sum_\gamma \lambda_\gamma \{\phi_\gamma^{(0)}, \phi_\alpha^{(0)}\} \ . \tag{6.11}$$

Define the structure functions $C_{\alpha\beta\gamma}^{[\ell\ell' k]}$ by

$$\{\phi_\alpha^{(\ell)}, \phi_\beta^{(\ell')}\} = \sum_{\gamma,k} C_{\alpha\beta\gamma}^{[\ell\ell' k]} \phi_\gamma^{(k)} \ . \tag{6.12}$$

They are the U_{AB}^C's defined in (5.12) written in the new notation. Then (6.11) becomes

$$\phi_\alpha^{(1)} = \vec{v}^{(1)}(\alpha) \cdot \vec{E}^{(1)} + \sum_{\beta,\gamma} \lambda_\gamma C_{\gamma\alpha\beta}^{[000]} (\vec{v}^{(0)}(\beta) \cdot \vec{E}^{(0)}) \, , \tag{6.13}$$

where use has been made of (6.6). Note that $\phi_\alpha^{(1)}$ is again a function of only q and p.

We now repeat the process and adjoin the time derivative of the constraint (6.13) to the equations of motion to construct $\vec{E}^{(2)}$:

$$\vec{E}^{(2)} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\vec{\phi}^{(0)} \\ \frac{d}{dt}\vec{\phi}^{(1)} \end{pmatrix} .$$

This leads to a matrix $\mathbf{W}^{(2)}$:

$$E_{a_2}^{(2)} = \sum_a W_{a_2 a}^{(2)} \dot{Q}_a + K_{a_2}^{(2)}(Q) \ .$$

As we continue with this iterative process, the number of new zero modes generated at each new level will in general be reduced, as "gauge identities" are being generated along the way (see below). Hence the number of components of $\vec{\phi}^{(\ell)}$ will in general decrease, as the level ℓ increases.

[5] If this is not the case then (6.9) determines one of the λ_γ's as a function of the rest, and the algorithm stops.

The constraints $\phi_\alpha^{(\ell)}$ with $\ell \geq 1$ can be iteratively constructed from the recursion relation,

$$\phi_\alpha^{(\ell+1)} = \vec{v}^{(\ell+1)}(\alpha) \cdot \vec{E}^{(\ell+1)} + \sum_{\ell'=0}^{\ell} \sum_{\beta,\gamma} \lambda_\gamma C_{\gamma\alpha\beta}^{[0\ell\ell']} \phi_\beta^{(\ell')} \,, \quad \ell \geq 0 \,, \tag{6.14}$$

where

$$\phi_\alpha^{(\ell+1)} := \{\phi_\alpha^{(\ell)}, H\} \,, \tag{6.15}$$

and the sum over β in (6.14) runs over all constraints $\phi_\beta^{(\ell')}$ at level ℓ'. The zero modes at level $\ell + 1$ have the following generic form :

$$\vec{v}^{(\ell+1)}(\beta) = (-\tilde{\nabla}\phi_\beta^{(\ell)}, \nabla\phi_\beta^{(\ell)}, \vec{0}, \hat{e}^{(\ell)}(\beta)) \,, \quad \ell \geq 0 \,. \tag{6.16}$$

Here $\hat{e}^{(\ell)}(\beta)$ is an $N_0 + N_1 + \cdots + N_\ell$ component unit vector with the only non-vanishing component at the position of the constraint $\phi_\beta^{(\ell)}$ in the array $(\vec{\phi}^{(0)}, \vec{\phi}^{(1)}, \cdots, \vec{\phi}^{(\ell)})$. The iterative process in a chain labeled by "α" will come to a halt at level $\ell = L_\alpha + 1$, when

$$\phi_\alpha^{(L_\alpha+1)} = \{\phi_\alpha^{(L_\alpha)}, H\} = \sum_{\ell=0}^{L_\alpha} \sum_\beta h_{\alpha\beta}^{[L_\alpha \ell]} \phi_\beta^{(\ell)} \,. \tag{6.17}$$

Making use of (6.14) and (6.17), and setting $\ell = L_\alpha$ in (6.14), this equation takes the form

$$G_\alpha := \vec{v}^{(L_\alpha+1)}(\alpha) \cdot \vec{E}^{(L_\alpha+1)} - \sum_{\ell=0}^{L_\alpha} \sum_\beta K_{\alpha\beta}^{[L_\alpha \ell]} \phi_\beta^{(\ell)} \equiv 0 \,, \tag{6.18}$$

where

$$K_{\alpha\beta}^{[L_\alpha \ell]} = h_{\alpha\beta}^{[L_\alpha \ell]} - \sum_\gamma \lambda_\gamma C_{\gamma\alpha\beta}^{[0 L_\alpha \ell]} \,. \tag{6.19}$$

Equation (6.18) expresses the fact, that the ϕ-chain labeled by "α" ends in a "gauge identity" at the level $L_\alpha + 1$. Since each chain must end in a gauge identity, there will be as many gauge identities as there are primary constraints. [6]

Iteration of (6.14), starting with $\ell = 0$, allows us to express all the constraints in terms of scalar products $\vec{v}^{(\ell)} \cdot \vec{E}^{(\ell)}$. Substituting the resulting expressions into (6.18), and multiplying each of the gauge identities $G_\alpha \equiv 0$ by an arbitrary function of time $u_\alpha(t)$, the content of all the identities can be summarized by an equation of the form

$$\sum_{\alpha=1}^{N_0} \sum_{\ell=0}^{L_\alpha+1} \rho_\alpha^{(\ell)}(Q, u) \left(\vec{v}^{(\ell)}(\alpha) \cdot \vec{E}^{(\ell)}\right) \equiv 0 \,, \tag{6.20}$$

[6] Recall that we have restricted ourselves to purely first class systems.

where

$$\vec{\mathbf{E}}^{(\ell)} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\vec{\phi}^{(0)} \\ \frac{d}{dt}\vec{\phi}^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ \frac{d}{dt}\vec{\phi}^{(\ell-1)} \end{pmatrix} . \tag{6.21}$$

Now, because of the generic structure of the eigenvectors (6.16) we have from (6.21)

$$\vec{v}^{(\ell+1)}(\alpha) \cdot \vec{E}^{(\ell+1)} = \sum_{a=1}^{2n} \left(v_a^{(\ell+1)}(\alpha) E_a^{(0)} \right) + \frac{d\phi_\alpha^{(\ell)}}{dt} , \quad \ell = 0, \cdots, L_\alpha , \tag{6.22}$$

where n is the number of coordinate degrees of freedom, and the constraints $\phi_\alpha^{(\ell)}$ appearing on the rhs can be expressed, by iterating (6.14), in terms of scalar products $\vec{v}^{(k)} \cdot \vec{E}^{(k)}$, which in turn can be decomposed in the form (6.22). Upon making a sufficient number of "partial decompositions" $udv = d(uv) - vdu$, the identity (6.20) can be written in the form

$$\sum_{\alpha=1}^{N_0} \sum_{\ell=0}^{L_\alpha} \epsilon_\alpha^{(\ell)} \sum_{a=1}^{2n+N_0} \left(v_a^{(\ell+1)}(\alpha) E_a^{(0)} \right) + \sum_{\alpha=1}^{N_0} \tilde{\epsilon}_\alpha \sum_{a=1}^{2n+N_0} (v_a^{(0)}(\alpha) E_a^{(0)}) - \frac{dF}{dt} \equiv 0 , \tag{6.23}$$

where $\epsilon_\alpha^{(\ell)}$ and $\tilde{\epsilon}_\alpha$ depend on the N_0 arbitrary functions of time $\{u_\alpha(t)\}$, as well as on the Q_a's and time derivatives thereof. This expression is of the form [7]

$$- \sum_{a=1}^{2n+N_0} E_a^{(0)} \delta Q_a = \frac{dF}{dt} , \tag{6.24}$$

where

$$\delta Q_a = - \sum_{\alpha=1}^{N_0} \sum_{\ell=0}^{L_\alpha} \epsilon_\alpha^{(\ell)} v_a^{(\ell+1)}(\alpha) - \sum_{\alpha=1}^{N_0} \tilde{\epsilon}_\alpha v_a^{(0)}(\alpha) . \tag{6.25}$$

For infinitesimal $\epsilon_\alpha^{(\ell)}$ the time integral of the lhs of (6.24) is just the variation (6.5) of the total action. Hence we conclude that the transformations (6.25) leave the total action (6.4) invariant. But because of the generic structure of the eigenvectors (6.16) we have from (6.25),

$$\delta q_i = \delta Q_i = \sum_{\alpha=1}^{N_0} \sum_{\ell=0}^{L_\alpha} \epsilon_\alpha^{(\ell)} \frac{\partial \phi_\alpha^{(\ell)}}{\partial p_i} = \sum_{\alpha=1}^{N_0} \sum_{\ell=0}^{L_\alpha} \epsilon_\alpha^{(\ell)} \{q_i, \phi_\alpha^{(\ell)}\} , \quad i = 1, \cdots, n , \tag{6.26}$$

[7] The minus sign has been introduced to cast the transformation laws in a standard form, when written in terms of Poisson brackets.

and

$$\delta p_i = \delta Q_{n+i} = -\sum_{\alpha=1}^{N_0}\sum_{\ell=0}^{L_\alpha} \epsilon_\alpha^{(\ell)} \frac{\partial \phi_\alpha^{(\ell)}}{\partial q_i} = \sum_{\alpha=1}^{N_0}\sum_{\ell=0}^{L_\alpha} \epsilon_\alpha^{(\ell)} \{p_i, \phi_\alpha^{(\ell)}\} \,, \tag{6.27}$$

$$\delta\lambda^\alpha = \delta Q_{2n+\alpha} = -\tilde{\epsilon}_\alpha \,, \tag{6.28}$$

where, as we have seen, the $\epsilon_\alpha^{(\ell)}$'s and $\tilde{\epsilon}_\alpha$ depend on N_0 arbitrary functions of time. Hence we conclude that all first class constraints act as generators of a local symmetry of the total action, and that the transformation of the coordinates and momenta are of the form (5.4) with the first class constraints generated in an iterative way according to (6.15). We have thus confirmed a proposition made in [Di Stefano 1983] regarding the form of the first class constraints that are generators of local symmetries.

Finally we would like to emphasize, that we have implicitly asumed throughout that the structure functions appearing in equations (6.17) and (6.18) are finite on the constraint surface. If this is not the case, we expect that our algorithm still generates the correct local *off-shell* symmetries of the total action if the gauge identities are non-singular on the constrained surface, but does not generate the symmetries on the level of the Hamilton equations of motion. This will be demonstrated in section 4. In the following section we first illustrate our formalism for the general case of two primary constraints and one secondary constraint.

6.3 General system with two primaries and one secondary constraint

In this section we apply the formalism to the case where the system exhibits two primary constraints $\phi_\alpha^{(0)}, \alpha = 1, 2$, and one secondary constraint $\phi_2^{(1)}$, which in Dirac's language is generated from the presistency in time of $\phi_2^{(0)}$. (This is illustrated by the table below.) The specific form of the Hamiltonian is irrelevant. *The constraints are assumed to be first class.*

$\phi_1^{(0)}$	$\phi_2^{(0)}$
	$\phi_2^{(1)}$

The total Lagrangian L_T is given by (6.1) with $N_0 = 2$. Proceeding as in section 2, the primary constraints can be written in the form (6.6), where $\vec{v}^{(0)}(1) = (\vec{0}, \vec{0}, 1, 0)$ and $\vec{v}^{(0)}(2) = (\vec{0}, \vec{0}, 0, 1)$.

Next we construct the vector (6.7), i.e. $\vec{E}^{(1)} = (\vec{E}^{(0)}, \dot{\phi}_1^{(0)}, \dot{\phi}_2^{(0)})$, and the zero modes of the corresponding matrix $\mathbf{W}^{(1)}$, which are given by (6.8), i.e.,

$\vec{v}^{(1)}(1) = (-\tilde{\nabla}\phi_1^{(0)}, \nabla\phi_1^{(0)}, 0, 0, 1, 0)$, $\vec{v}^{(1)}(2) = (-\tilde{\nabla}\phi_2^{(0)}, \nabla\phi_2^{(0)}, 0, 0, 0, 1)$. Since only $\phi_2^{(0)}$ is assumed to lead to a new constraint, we are immediately left with one gauge identity at level 1, generated from $\phi_1^{(0)}$:

$$G_1 = \vec{v}^{(1)}(1) \cdot \vec{E}^{(1)} - \sum_{\beta=1}^{2} K_{1\beta}^{[00]} \phi_\beta^{(0)} \equiv 0\,,$$

or

$$G_1 = \vec{v}^{(1)}(1) \cdot \vec{E}^{(1)} - \sum_{\beta=1}^{2} K_{1\beta}^{[00]} \left(\vec{v}^{(0)}(\beta) \cdot \vec{E}^{(0)}\right) \equiv 0\,, \tag{6.29}$$

which is nothing but (6.18) with $L_1 = 0$. On the other hand $\phi_2^{(0)}$ gives rise, by assumption, to a new constraint $\phi_2^{(1)}$ at level 1, which is given by

$$\phi_2^{(1)} = \vec{v}^{(1)}(2) \cdot \vec{E}^{(1)} + \sum_{\beta,\gamma=1}^{2} \lambda_\gamma C_{\gamma 2\beta}^{[000]} \left(\vec{v}^{(0)}(\beta) \cdot \vec{E}^{(0)}\right)\,.$$

We are therefore led to construct

$$\vec{E}^{(2)} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}\phi_1^{(0)} \\ \frac{d}{dt}\phi_2^{(0)} \\ \frac{d}{dt}\phi_2^{(1)} \end{pmatrix}\,,$$

as well as the corresponding rectangular matrix $W^{(2)}$ and its left zero modes, of which only the contraction of

$$\vec{v}^{(2)}(2) = (-\tilde{\nabla}\phi_2^{(1)}, \nabla\phi_2^{(1)}, 0, 0, 0, 0, 1)$$

with $\vec{E}^{(2)}$ leads to a new equation, which is necessarily a gauge identity, since we have assumed that the system only possesses one secondary constraint $\phi_2^{(1)}$. The gauge identity at level 2 has the form

$$G_2 = \vec{v}^{(2)}(2) \cdot \vec{E}^{(2)} - \sum_{\beta=1}^{2} K_{2\beta}^{[10]} \phi_\beta^{(0)} - K_{22}^{[11]} \phi_2^{(1)} \equiv 0$$

or

$$\begin{aligned} G_2 = {} & \vec{v}^{(2)}(2) \cdot \vec{E}^{(2)} - \sum_{\beta=1}^{2} K_{2\beta}^{[10]} \left(\vec{v}^{(0)}(\beta) \cdot \vec{E}^{(0)}\right) \\ & - K_{22}^{[11]} \left(\vec{v}^{(1)}(2) \cdot \vec{E}^{(1)} + \sum_{\beta,\gamma=1}^{2} \lambda_\gamma C_{\gamma 2\beta}^{[000]} \left(\vec{v}^{(0)}(\beta) \cdot \vec{E}^{(0)}\right) \right) \equiv 0\,. \end{aligned} \tag{6.30}$$

Multiplying the gauge identities (6.29) and (6.30) by the arbitrary functions $u_1(t)$ and $u_2(t)$, respectively, and taking their sum, the resulting expression can be written in the form (6.20). Upon making a sufficient number of "partial differential decompositions", one finds, after some algebra, that the information encoded in the gauge identities can be written in the form (6.23), where

$$\begin{aligned}
\epsilon_1^{(0)} &= -u_1 \\
\epsilon_2^{(1)} &= -u_2 \\
\epsilon_2^{(0)} &= \dot{u}_2 + u_2 K_{22}^{[11]} \\
\delta\lambda_\alpha &= -\dot{u}_2 \sum_\gamma \lambda_\gamma C_{\gamma 2\alpha}^{[000]} - u_2 K_{2\alpha}^{[10]} - u_1 K_{1\alpha}^{[00]} \\
&\quad - u_2 K_{22}^{[11]} \sum_\gamma \lambda_\gamma C_{\gamma 2\alpha}^{[000]} + \ddot{u}_2 \delta_{2\alpha} - \dot{u}_1 \delta_{1\alpha} + \frac{d}{dt}\left(u_2 K_{22}^{[11]}\right) \delta_{2\alpha}
\end{aligned}$$

are the only non-vanishing parameters. The corresponding transformation laws for the coordinates q_i and momenta p_i are given by (6.26) and (6.27).

Having obtained $\epsilon_1^{(0)}, \epsilon_2^{(0)}, \epsilon_2^{(1)}$ and $\delta\lambda^\alpha$ expressed in terms of $u_1(t)$ and $u_2(t)$, one now verifies that the $\epsilon_\alpha^{(\ell)}$ are solutions to the recursion relations following from the requirement that the transformation laws (6.26) and (6.27) be symmetries of the total action. In the present notation where the label A of ϵ^A in (5.14) is replaced by the pair (ℓ, α), labeling the level and (parent) primary constraint, the relations (5.14) and (5.15) take the form

$$\frac{d\epsilon_\alpha^{(\ell)}}{dt} + \sum_{\ell'=0} \sum_\beta \epsilon_\beta^{(\ell')} K_{\beta\alpha}^{[\ell'\ell]} = 0\,, \quad \ell \geq 1$$

and

$$\delta\lambda_\alpha = \dot{\epsilon}_\alpha^{(0)} + \sum_{\ell'=0} \sum_\beta \epsilon_\beta^{(\ell')} K_{\beta\alpha}^{[\ell'0]}\,. \tag{6.31}$$

In our case these equations reduce to

$$\frac{d\epsilon_2^{(1)}}{dt} + \sum_{\beta=1}^{2} \epsilon_\beta^{(0)} K_{\beta 2}^{[01]} + \epsilon_2^{(1)} K_{22}^{[11]} = 0\,, \tag{6.32}$$

and

$$\delta\lambda_\alpha = \dot{\epsilon}_\alpha^{(0)} - \sum_\beta \epsilon_\beta^{(0)} K_{\beta\alpha}^{[00]} - \epsilon_2^{(1)} K_{2\alpha}^{[10]} = 0\,, \tag{6.33}$$

where we have made use of the fact that, because of the way in which the constraints have been generated, $K_{\alpha 1}^{[\ell 1]} = 0$ $(\alpha = 1, 2)$, $K_{22}^{[01]} = 1$, $K_{12}^{[01]} = 0$, $C_{\alpha\beta\gamma}^{[001]} = 0$, and $K_{22}^{[00]} = -\sum_\gamma \lambda^\gamma C_{\gamma 22}^{[000]}$. Substituting (6.31) into (6.32) and (6.33) one then verifies that the above equations are indeed satisfied.

6.4 Counterexamples to Dirac's conjecture?

In the literature it has been stated that Dirac's conjecture is not always correct [Cawley 1979, Henneaux 1992, Miscovic 2003]. In the following we present three models which have served as examples for this and show, that there is no clash with Dirac's conjecture when interpreted as described in section 2.

Example 1

An example considered in [Henneaux 1992] is given by the Lagrangian

$$L = \frac{1}{2} e^{q_2} \dot{q}_1^2 \,. \tag{6.34}$$

In the following we analyze this system in detail, and point out some subtleties leading to an apparent clash with Dirac's conjecture.

The Lagrange equations of motion can be summarized by a single equation

$$\dot{q}_1 = 0 \,. \tag{6.35}$$

Hence q_2 is an arbitrary function.

Equation (6.35) does not possess a local symmetry. On the Hamiltonian level the system nevertheless exhibits two first class constraints, which, as we now show induce transformations which are off-shell symmetries of the total action $\int dt\ L_T$, with L_T defined in (6.1). However only one of the constraints generates a symmetry of the Hamilton equations of motion.

From (6.34) we obtain for the (only) primary constraint

$$\phi^{(0)} = p_2 = 0 \,. \tag{6.36}$$

The canonical Hamiltonian evaluated on the primary surface is given by

$$H_0(q,p) = \frac{1}{2} e^{-q_2} p_1^2 \,,$$

and the first order total Lagrangian reads

$$L_T(q,p,\lambda;\dot{q},\dot{p},\dot{\lambda}) = \sum_i p_i \dot{q}_i - H_0(q,p) - \lambda \phi^{(0)} \,.$$

Considered as a function of the "coordinates" q_i, p_i, λ and their time derivatives, it has the form (6.2) with $2n + N_0 = 5$, $Q_a = (\vec{q}, \vec{p}, \lambda)$, and $a_i = p_i$ $(i = 1, 2)$ for the non-vanishing components of a_a.

The Euler derivatives are given by (6.3), where $W^{(0)}_{ab} = \partial_a a_b - \partial_b a_a$ are the elements of a 5×5 matrix with non-vanishing components $W_{13} = W_{24} =$

$-W_{31} = -W_{42} = -1$. The equations of motion are given by $E_a^{(0)} = 0$, and yield the Hamilton equations of motion

$$\begin{aligned} &\dot{q}_1 - e^{-q_2} p_1 = 0\ , \quad \dot{q}_2 - \lambda = 0\ , \\ &\dot{p}_1 = 0\ , \quad \dot{p}_2 - \frac{1}{2} e^{-q_2} p_1^2 = 0\ , \end{aligned} \tag{6.37}$$

as well as the constraint (6.36). We now proceed with the construction of the constraints and gauge identities as described in section 2. Since we have only one primary constraint, the formalism simplifies considerably.

The matrix $W_{ab}^{(0)}$ has one left zero-mode $\vec{v}^{(0)} = (0, 0, 0, 0, 1)$, whose contraction with $\vec{E}^{(0)}$ just reproduces on shell the primary constraint: $\vec{v}^{(0)} \cdot \vec{E}^{(0)} = \phi^{(0)}$. Proceeding in the manner described in section 2, we construct $\vec{E}^{(1)}$ and the corresponding left eigenvector of $W^{(1)}$, $v^{(1)} = (0, -1, 0, 0, 0, 1)$, leading to the secondary constraint $\vec{v}^{(1)} \cdot \vec{E}^{(1)} = \phi^{(1)}$, where

$$\vec{E}^{(1)} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d\phi^{(0)}}{dt} \end{pmatrix} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}(\vec{v}^{(0)} \cdot \vec{E}^{(0)}) \end{pmatrix} , \tag{6.38}$$

and

$$\phi^{(1)} = \frac{1}{2} e^{-q_2} p_1^2 \, . \tag{6.39}$$

Note that by construction $\phi^{(1)} = 0$ on shell, as seen from (6.36) and (6.37). Since $\{\phi^{(0)}, \phi^{(1)}\} = \phi^{(1)}$, the constraints $\phi^{(0)}$ and $\phi^{(1)}$ form a first class system. The algorithm is found to stop at level “2” where the following gauge identity is generated,

$$\vec{v}^{(2)} \cdot \vec{E}^{(2)} + \lambda \vec{v}^{(1)} \cdot \vec{E}^{(1)} \equiv 0\, ,$$

with

$$\vec{v}^{(2)} = (-e^{-q_2} p_1, 0, 0, -\frac{1}{2} e^{-q_2} p_1^2, 0, 0, 1)\, ,$$

and

$$\vec{E}^{(2)} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d\phi^{(0)}}{dt} \\ \frac{d\phi^{(1)}}{dt} \end{pmatrix} = \begin{pmatrix} \vec{E}^{(0)} \\ \frac{d}{dt}(\vec{v}^{(0)} \cdot \vec{E}^{(0)}) \\ \frac{d}{dt}(\vec{v}^{(1)} \cdot \vec{E}^{(1)}) \end{pmatrix} . \tag{6.40}$$

The gauge identity can be reduced to the form

$$\begin{aligned} \sum_{a=1}^{5} \Big[v_a^{(2)} E_a^{(0)} &+ \frac{d}{dt}(v_a^{(1)} E_a^{(0)}) + \frac{d^2}{dt^2}(v_a^{(0)} E_a^{(0)}) \\ &+ \lambda [(v_a^{(1)} E_a^{(0)}) + \frac{d}{dt}(v_a^{(0)} E_a^{(0)})] \Big] \equiv 0\, . \end{aligned}$$

Multiplying this expression by an arbitrary function of time $\epsilon(t)$, this identity becomes of the form (6.24), with

$$\delta Q_a = \epsilon v_a^{(2)} + (\lambda\epsilon - \dot{\epsilon})v_a^{(1)} + \left(\ddot{\epsilon} - \frac{d}{dt}(\lambda\epsilon)\right) v_a^{(0)} . \tag{6.41}$$

In terms of the Hamiltonian variables, q_i, p_i and λ, (6.41) implies the following transformation laws

$$\begin{aligned}
\delta q_1 &= -\epsilon e^{-q_2} p_1 = \epsilon^{(1)}\{q_1, \phi^{(1)}\} \,, \\
\delta q_2 &= = \dot{\epsilon} - \lambda\epsilon = \epsilon^{(0)}\{q_2, \phi^{(0)}\} \,, \\
\delta p_1 &= 0 \,, \\
\delta p_2 &= -\frac{1}{2}\epsilon e^{-q_2} p_1^2 = \epsilon^{(1)}\{p_2, \phi^{(1)}\} \,, \\
\delta\lambda &= \ddot{\epsilon} - \frac{d}{dt}(\lambda\epsilon) = \dot{\epsilon}^{(0)} \,,
\end{aligned} \tag{6.42}$$

where

$$\epsilon^{(0)} = \dot{\epsilon} - \lambda\epsilon \,, \quad \epsilon^{(1)} = -\epsilon \,.$$

One readily verifies, that the $\epsilon^{(\ell)}$ satisfy the recursion relations (5.14). We have thus verified that the transformations generated by the first class constraints (6.36) and (6.39), *obtained iteratively in the systematic way according to (6.15)*, do indeed generate a symmetry of the action. This symmetry is realized *off shell* and requires the full set of transformation laws (6.42). On the other hand, the (also first class) constraint $\tilde{\phi}^{(1)} := p_1 = 0$, although formally equivalent to the secondary constraint (6.39), is not a generator of a local symmetry of the total action. Indeed, as observed in [Henneaux/Teitelboim 1992], p_1 induces translations in q_1, which is not a symmetry of the total Lagrangian. Note further that on the constrained surface, the variations of δq_1 and δp_2 vanish. Thus $\phi^{(1)}$ becomes an "ineffective" generator on shell. The remaining symmetry is just the statement $\delta\lambda = \frac{d}{dt}\delta q_2$, which is consistent with (6.37), i.e., with $\delta\dot{q}_2 = \delta\lambda$. It has no analogue on the level of the Lagrange equations of motion following from (6.34).

Example 2

The following example taken from [Miskovic 2003] illustrates our comment made at the end of section 2 regarding the case where the structure functions (6.19) are singular on the constrained surface, while the gauge identitites are well defined on that surface.

Consider the Lagrangian,

$$L = \frac{1}{2}\dot{q}^2 + uf^k(q) \,, \quad k > 1 \,,$$

with $f(q)$ analytic in the neighborhood of $f(q) = 0$, which has been classified as being of "type II" in [Miskovic 2003]. [8] The canonical Hamiltonian evaluated on the primary surface $p_u = 0$ is given by $H_0 = \frac{1}{2}p^2 - uf^k(q)$, and correspondingly the first order Lagrangian reads

$$L_T(Q, \dot{Q}) = p\dot{q} + p_u\dot{u} - \frac{1}{2}p^2 + uf^k(q) - \lambda p_u \,,$$

where $Q = (q, u, p, p_u, \lambda)$. The Euler-Lagrange equations derived from L_T are just the Hamilton equations of motion. Following the Lagrangian algorithm of section 2 we obtain for the left zero modes of the matrix $W^{(\ell)}$, at levels $\ell = 0, 1, 2$,

$$\begin{aligned}
\vec{v}^{(0)} &= (0, 0, 0, 0, 1) \\
\vec{v}^{(1)} &= (0, -1, 0, 0, 0, 1) \,, \\
\vec{v}^{(2)} &= (0, 0, f^{k-1}f', 0, 0, 0, 1) \,,
\end{aligned}$$

leading to the constraints

$$\begin{aligned}
\phi^{(0)} &:= \vec{v}^{(0)} \cdot \vec{E}^{(0)} = p_u = 0 \,, \\
\phi^{(1)} &:= \vec{v}^{(1)} \cdot \vec{E}^{(1)} = f^k(q) = 0 \,,
\end{aligned}$$

and the gauge identity

$$\vec{v}^{(2)} \cdot \vec{E}^{(2)} - kp(\ln f)'\vec{v}^{(1)} \cdot \vec{E}^{(1)} = 0 \,, \tag{6.43}$$

where the prime denotes the derivative with respect to q. Since $k > 1$, the second term on the lhs of (6.43) is well defined on the constrained surface $\vec{v}^{(1)} \cdot \vec{E}^{(1)} = f^k(q) = 0$, and in fact vanishes there. The "coordinate" dependent factor $kp(\ln f)'$ in (6.43) is nothing but the structure function K in (6.19). This function is singular on the constrained "surface" $f(q) = 0$, whereas the gauge identity (6.43) is finite there and "generates" *off-shell* the correct local symmetry transformation of the total action, as we now demonstrate.

In the present example $\vec{E}^{(1)}$ and $\vec{E}^{(2)}$ have again the form of (6.38) and (6.40), respectively. Making use of these relations, and multiplying the gauge identity (6.43) by $\epsilon(t)$, one finds that the resulting identity can be written in the form (6.24) with δQ_a the infinitesimal transformations,

$$\begin{aligned}
\delta\, q &= 0 \,, \\
\delta\, u &= \dot{\epsilon} + \epsilon kp(\ln f)' \,, \\
\delta\, p &= \epsilon k f^{k-1} f' \,, \\
\delta\, p_u &= 0 \,, \\
\delta\, \lambda &= \ddot{\epsilon} + \frac{d}{dt}(\epsilon kp(\ln f)') \,.
\end{aligned} \tag{6.44}$$

[8] In order to simplify the discussion we have restricted ourselves to two degrees of freedom, q and u.

These transformations are only defined off-shell. Computing the corresponding variation δL_T one finds that it is given by a total time derivative

$$\delta L_T = \frac{d}{dt}(\epsilon f^k(q)) .$$

In accordance with our expectations, one verifies that δQ_a can be written in the form

$$\delta Q_a = \sum_{\ell=0}^{1} \epsilon_\ell \{Q_a, \phi^{(\ell)}\} , \quad a = 1, \cdots, 4 ,$$

where

$$\epsilon_0 = \dot{\epsilon} + \epsilon k p(\ln f)' ,$$
$$\epsilon_1 = -\epsilon .$$

For the Lagrange multiplier one has $\delta\lambda = \frac{d}{dt}\epsilon_0$.

The infinitesimal transformations (6.44) represent symmetry transformations of the total action, away from the constraint surface $f^k(q) = 0$. They are not defined on the surface $f^k(q) = 0$. Correspondingly they do not represent symmetries of the Hamilton equations of motion.

Example 3: bifurcations

We now consider an example of a system exhibiting bifurcations of constraints, such as considered in [Lusanna 1991], and show that also in this case our algorithm generates correctly the gauge symmetries of the total action, whereas the first class constraints corresponding to a particular branch of these bifurcations - which are not part of our iterative algorithm - do not generate gauge symmetries of the total action. Consider the so-called "Christ-Lee model" [Christ 1980] discussed in [Lusanna 1991], and defined by the Lagrangian

$$L = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 - q_3(q_1\dot{q}_2 - q_2\dot{q}_1) - V(q_1^2 + q_2^2) + \frac{1}{2}q_3^2(q_1^2 + q_2^2) ,$$

with $V(x)$ some potential. The equivalent first order Lagrangian in our approach reads,

$$L_T = \sum_i p_i\dot{q}_i - \frac{1}{2}p_1^2 - \frac{1}{2}p_2^2 - q_3(q_1p_2 - q_2p_1) - V(q_1^2 + q_2^2) - \lambda p_3 .$$

The Euler-Lagrange equations take the form $E_a^{(0)}(Q, \dot{Q}) = 0$, where

$$E_1^{(0)} = \dot{p}_1 + p_2q_3 + \partial_1 V , \quad E_2^{(0)} = \dot{p}_2 - p_1q_3 + \partial_2 V ,$$
$$E_3^{(0)} = \dot{p}_3 + (q_1p_2 - q_2p_1) , \quad E_4^{(0)} = -\dot{q}_1 + p_1 - q_2q_3 ,$$
$$E_5^{(0)} = -\dot{q}_2 + p_2 + q_1q_3 , \quad E_6^{(0)} = -\dot{q}_3 + \lambda , \quad E_7^{(0)} = p_3 ,$$

and $Q_a : (q_1, q_2, q_3, p_1, p_2, p_3, \lambda)$. Going through the procedure of section 2 we arrive at a level zero and level one constraint,

$$\phi^{(0)} := \vec{v}^{(0)} \cdot \vec{E}^{(0)} = p_3 = 0\,, \quad \phi^{(1)} := \vec{v}^{(1)} \cdot \vec{E}^{(1)} = q_2 p_1 - q_1 p_2 = 0\,,$$

where $\vec{v}^{(0)} = (0,0,0,0,0,0,1)$ and $\vec{v}^{(1)} = (0,0,-1,0,0,0,0,1)$, and $\vec{E}^{(1)}$ has the form (6.38). The iterative process stops at level two, with the gauge identity $\vec{v}^{(2)} \cdot \vec{E}^{(2)} \equiv 0$, where $\vec{E}^{(2)}$ has the form (6.40) and with the level 2 zero mode given by $\vec{v}^{(2)} = (-q_2, q_1, 0, -p_2, p_1, 0, 0, 0, 1)$. The simple form of this identity reflects the fact that the structure functions $K^{[2\ell]}_{\alpha\beta}$ in (6.19) vanish for the case in question. Explicitly one has

$$\vec{v}^{(2)} \cdot \vec{E}^{(2)} = \sum_a \left[\left(\vec{v}^{(2)} \cdot \vec{E}^{(0)} \right) + \frac{d}{dt} \left(\vec{v}^{(1)} \cdot \vec{E}^{(0)} \right) + \frac{d^2}{dt^2} \left(\vec{v}^{(0)} \cdot \vec{E}^{(0)} \right) \right] \equiv 0. \tag{6.45}$$

This equation, when multiplied by $\epsilon(t)$, can be written in the form (6.24) with $F = \dot{\epsilon}\phi^{(0)} - \epsilon\phi^{(1)}$. The $\{\delta Q_a\}$ can be compactly written as

$$\delta Q_a = \sum_{\ell=0}^{1} \epsilon_\ell \{Q_a, \phi^{(\ell)}\}\,, \quad \delta\lambda = \ddot{\epsilon}\,,$$

where $\epsilon_0 = \dot{\epsilon}$, $\epsilon_1 = -\epsilon$.

The constraints $\phi^{(0)} = 0$ and $\phi^{(1)} = 0$ are first class and are thus found to generate a gauge symmetry of the total action, in agreement with Dirac's conjecture.

Suppose now that we had chosen instead of $\phi^{(1)} = 0$ any one of the following bifurcations: i) $q_1 = q_2 = 0$, ii) $p_1 = p_2 = 0$, iii) $q_1 = p_1 = 0$, and iv) $q_2 = p_2 = 0$. Consider for instance the case i). [9] Then we continue at level one with the constraints, $\bar{\phi}_1^{(1)} := q_1 = 0 \quad \bar{\phi}_2^{(1)} := q_2 = 0$. The requirement of persistence in time then leads to two new constraints at level two: $\bar{\phi}_1^{(2)} = p_1 = 0\,, \quad \bar{\phi}_2^{(2)} := p_2 = 0$. Since $\dot{\bar{\phi}}_1^{(2)} \approx 0$ and $\dot{\bar{\phi}}_2^{(2)} \approx 0$ there are no further constraints. We thus see that the system of constraints has been reduced to a mixed system with two second class constraints, and just one first class constraint $\phi^{(0)} := p_3 = 0$. Correspondingly we are led to consider $\bar{G} = \epsilon p_3$ as a potential generator for a local symmetry. One readily verifies that all variations vanish except for δq_3 and $\delta\lambda$. Hence

$$\delta L_T = (\dot{\epsilon} - \delta\lambda)p_3 - \epsilon(q_1 p_2 - q_2 p_1) = (\dot{\epsilon} - \delta\lambda)\phi^{(0)} + \epsilon\phi^{(1)}\ .$$

To have a symmetry we must demand that $\delta\lambda = \dot{\epsilon}$ and $\phi^{(1)} = 0$, so that L_T only exhibits a local invariance on the restricted surface defined by the *first*

[9]We are leaving herewith our algorithm, since with the choice of a particular branch the constraint can no longer be written in the form $\vec{v}^{(1)} \cdot \vec{E}^{(1)} = 0$.

class constraint $\phi^{(1)} = 0$, generated at level 1 by the algorithm of section 2, i.e. $\bar{G}$ does not generate an off-shell symmetry of the total action.

Summarizing, our examples have elucidated several important points: *i)* The replacement of constraints, generated iteratively by our algorithm, by a *formally* equivalent set of constraints is, in general, not allowed, and in fact may obliterate the full symmetry of the total action. That the validity of Dirac's conjecture depends crucially on the chosen form for the constraints has already been noted in [Di Stefano 1983]. *ii)* The second example illustrated that our algorithm may generate correctly the symmetries of the total action away from the constrained surface, although the structure function K is singular on that surface. The gauge identity was however well defined everywhere. Finally, example 3 illustrated that in the case of bifurcations of constraints, the choice of a particular branch may change the structure of the first class constraints, such as to be in conflict with Dirac's conjecture. We emphasize that such a procedure is excluded in our algorithm.

Chapter 7

BFT Embedding of Second Class Systems

7.1 Introduction

As we have remarked at various occations, the quantization of systems involving second class constraints may present serious problems on operator as well as functional level, because of the possible non-polynomial structure of the Dirac brackets involved. As we will see in the following chapter, the Hamilton-Jacobi formulation of second class or mixed systems is also problematic. On the other hand purely first class systems admit an elegant quantization with a canonical Poisson bracket structure, based on the existence of a nilpotent charge as generator of a BRST symmetry. Furthermore such systems also allow for a straightforward Hamilton-Jacobi formulation. It is therefore desirable to find a way of embedding a system with mixed first and second class constraints into a purely first class system.

In this chapter we will make use of a general canonical formalism developed by Batalin, Fradkin and Tyutin [Batalin 1987/91] for converting in a systematic way a pure second class system into a first class one, by increasing the number of degrees of freedom to include unphysical ones. The second class system can then be understood as a gauge theory in a particular gauge, and can be quantized along the lines to be discussed in chapters 11 and 12.

Since we shall deal with purely *second* class systems, we will simplify the notation by simply denoting the second class constraints $\Omega^{(2)}_{A_2}$ by Ω_α. The subindex has been chosen to facilitate a comparison with the literature, and does not follow our previous convensions, where the Greek indices were use to label the primary constraints.

7.2 Summary of the BFT-procedure

To begin with we provide the reader with a short summary of the basic procedure, given by Batalin, Fradkin and Tyutin (BFT), for converting a second class system to an equivalent first class system, where gauge invariant observables are identified in a unique way with corresponding observables in the original second class theory. The following discussion is carried out entirely on the classical level [Batalin 1991].

Let $\Omega_\alpha(q,p)$ denote the second class constraints associated with some Hamiltonian $H(q,p)$. Recall that the number of such constraints is always even. Let their number be N_Ω. Following BFT, we extend the space by introducing for each constraint Ω_α a field ϕ^α with Poisson brackets

$$\{\phi^\alpha, \phi^\beta\} = \omega^{\alpha\beta} \ . \tag{7.1}$$

The Poisson bracket of two functions $f(\phi)$ and $g(\phi)$ is then given by

$$\{f, g\} = \sum_{\alpha,\beta} \frac{\partial f}{\partial \phi^\alpha} \omega^{\alpha\beta} \frac{\partial g}{\partial \phi^\beta} \tag{7.2}$$

with $\omega^{\alpha\beta}$ a c-numerical matrix which is antisymmetric in the following sense

$$\omega^{\beta\alpha} = -(-1)^{\epsilon_\alpha \epsilon_\beta} \omega^{\alpha\beta} \ , \tag{7.3}$$

but can otherwise be freely chosen. Here ϵ_α is the Grassmann signature of ϕ^α. [1] The ϕ^α's have vanishing Poisson brackets with the remaining variables. Because of the simplectic structure of (7.1) one can always construct linear combinations $\theta^{\bar\alpha}$ and $p^\theta_{\bar\alpha}$ of the $\{\phi^\alpha\}$ such that the non-vanishing brackets take the canonical form.

$$\{\theta^{\bar\alpha}, p^\theta_{\bar\beta}\} = \delta^{\bar\alpha}_{\bar\beta} \ ,$$

where $\bar\alpha$, $\bar\beta$ run over half the number of values of α, β.

The next step consists in constructing a new set of constraints $\tilde\Omega_\alpha(q,p,\phi) = 0$ which are in strong involution,

$$\{\tilde\Omega_\alpha, \tilde\Omega_\beta\} = 0 \,, \tag{7.4}$$

and satisfy the boundary condition

$$\tilde\Omega_\alpha|_{\phi=0} = \Omega_\alpha \,. \tag{7.5}$$

[1] Equation (7.3) includes the case where the ϕ^α's are Grassmann valued. For bosonic (fermionic) variables the Grassmann signature is given by $\epsilon = 0$ ($\epsilon = 1$). In the following we will however consider purely bosonic systems.

Here, and in what follows, it will always be understood that Poisson brackets of tilde-variables are taken with respect to the extended phase space. A corresponding Lagrangian $\tilde{L}$ is not evoked at any stage. Whatever this Lagrangian may be, the constraints $\tilde{\Omega}_\alpha = 0$, following from it, being first class, generate a local symmetry of $\tilde{L}$. Hence (gauge invariant) observables $\tilde{A}$ in the extended system are required to satisfy

$$\{\tilde{A}, \tilde{\Omega}_\alpha\} = 0 \ , \tag{7.6}$$

together with

$$\tilde{A}(q, p, \phi)|_{\phi=0} = A(q, p) \ . \tag{7.7}$$

In particular, the Hamiltonian of the extended system must itself be gauge invariant and therefore satisfy [2]

$$\{\tilde{H}, \tilde{\Omega}_\alpha\} = 0 \ , \tag{7.8}$$

with

$$\tilde{H}(q, p, \phi)|_{\phi=0} = H(q, p) \ . \tag{7.9}$$

An observable $\tilde{A}$ of the embedded system which satisfies (7.6) and (7.7) is said to be the gauge invariant extension of A of the original system. This extension is not unique. Thus we can always add terms proportional to the first class constraints of the embedded system, since these are in strong involution.

We now prove a number of propositions which will play a central role in the discusssion to follow. But first a definition:

Definition: An embedding is called *locally regular* if there exists an open neighbourhood of $\{\phi^\alpha\} = 0$, where $\Omega_\alpha = \tilde{\Omega}_\alpha, \ \forall\alpha$ implies that $\phi^\alpha = 0, \ \forall\alpha$.

We then have

[2] In the BFT scheme the new Hamiltonian is also required to be in strong involution with the constraints. This must not be so. In [Shirzad 2004] an alternative scheme was proposed, where the new Hamiltonian is in weak involution with the constraints in a "chain" generated from a given primary constraint in a way smilar to (5.47), except for the last member in the chain, which is taken to be in strong involution with the Hamiltonian:

$$\begin{aligned}
&\{\tilde{\Omega}_\alpha, \tilde{\Omega}_\beta\} = 0 \\
&\{\tilde{H}, \tilde{\Omega}_\alpha\} = \tilde{\Omega}_{\alpha+1} \, , \quad \alpha = 1, \cdots, N-1 \\
&\{\tilde{H}, \tilde{\Omega}_N\} = 0 \, .
\end{aligned}$$

Proposition 1

An embedding where the new constraints satisfy (7.4) and (7.5), and are analytic in the auxiliary variables within a neighbourhood of $\phi = 0$, provides a locally regular embedding.

Indeed, because of (7.5) and the assumed analyticity property, we may write

$$\tilde{\Omega}_\alpha = \Omega_\alpha + X_{\alpha\gamma}\phi^\gamma + \cdots , \tag{7.10}$$

where the "dots" stand for terms of higher order in the auxiliary variables ϕ^α. The $X_{\alpha\gamma}$ may depend on q and p. Hence, because of (7.4)

$$\{\tilde{\Omega}_\alpha, \tilde{\Omega}_\beta\}|_{\phi=0} = Q_{\alpha\beta} + X_{\alpha\gamma}\omega^{\gamma\delta}X^T_{\delta\beta} = 0\,, \tag{7.11}$$

where $Q_{\alpha\beta}$ is the "Dirac" matrix defined by

$$\{\Omega_\alpha, \Omega_\beta\} = Q_{\alpha\beta}\,. \tag{7.12}$$

Since the matrices $Q_{\alpha\beta}$ and $\omega^{\alpha\beta}$ are invertible, the matrix $X_{\alpha\beta}$ is also invertible. Then (7.10) implies that $\phi^\alpha = X^{\alpha\beta}(\tilde{\Omega}_\beta - \Omega_\beta)$, where the matrix $X^{\alpha\beta}$ is the inverse of $X_{\alpha\beta}$. From this we conclude that proposition 1 holds.

As we have already pointed out, the gauge invariant extension of observables $A(q,p)$ of the second class system is not unique. Different extensions, satisfying (7.6), may differ, because of (7.4), by terms proportional to the first class constraints of the embedded system. This leads us to

Proposition 2

A particular gauge invariant extension of $A(q,p)$ can be obtained by simply replacing the q_i's and p_i's by their gauge invariant extensions, i.e. the $\tilde{q}_i$'s and $\tilde{p}_i$'s:

$$\tilde{A} = A(\tilde{q}, \tilde{p})\ .$$

The gauge invariance of $\tilde{A}$, i.e. $\{\tilde{A}, \tilde{\Omega}_\alpha\} = 0$, is then guaranteed by the chain rule

$$\{\tilde{A}, \tilde{\Omega}_\alpha\} = \sum_{i=1}^{n}\left[\frac{\partial \tilde{A}}{\partial \tilde{q}_i}\{\tilde{q}_i, \tilde{\Omega}_\alpha\} + \frac{\partial \tilde{A}}{\partial \tilde{p}_i}\{\tilde{p}_i, \tilde{\Omega}_\alpha\}\right]\ ,$$

where all Poisson brackets are computed in the extended space, making use of (7.2). This can be easily proven. The same expression holds for $\tilde{\Omega}_\alpha$ replaced by $\tilde{B}(\tilde{q}, \tilde{p})$. The advantage of this particular extension is that the embedding procedure need not be repeated for every observable separately, but need only be carried out once for the dynamical variables in question. Moreover, in cases where the expansion in the auxiliary variables terminates after the first order term, the form of the tilde-variables $\tilde{q}$ and $\tilde{p}$ can be frequently obtained by inspection, whereas this is not the case for the strongly involutive Hamiltonian $\tilde{H}$.

Proposition 3

For $\phi = 0$, the Poisson bracket of two observables $\tilde{A}(q,p,\phi)$ and $\tilde{B}(q,p,\phi)$, which are analytic in the neighbourhood of $\phi = 0$, equals the Dirac bracket of the corresponding observables A and B of the second class system

$$\{\tilde{A},\tilde{B}\}_{\phi=0} = \{A,B\}_D \,. \tag{7.13}$$

<u>*Proof*</u>

Let $\tilde{A} = A + a_\gamma \phi^\gamma + \cdots$, where the dots stand for higher orders in ϕ. Because of (7.10) we then have

$$\{\tilde{A},\tilde{\Omega}_\beta\}|_{\phi=0} = \{A,\Omega_\beta\} + a_\lambda \omega^{\lambda\rho} X_{\beta\rho} = 0\,,$$

or

$$a_\gamma = -\{A,\Omega_\alpha\} X^{\delta\alpha}\omega_{\delta\gamma}\ ,$$

where $\omega_{\alpha\beta}$ and $X^{\alpha\beta}$ are the inverse of the matrices $\omega^{\alpha\beta}$ and $X_{\alpha\beta}$:

$$\omega_{\alpha\gamma}\omega^{\gamma\beta} = \delta_\alpha^{\ \beta}\,, \quad X^{\alpha\gamma}X_{\gamma\beta} = \delta^\alpha_{\ \beta}\,.$$

Similar expressions hold with A replaced by B, and a_γ replaced by b_γ. Now

$$\{\tilde{A},\tilde{B}\}_{\phi=0} = \{A,B\} + a_\gamma\omega^{\gamma\delta}b_\delta\,,$$

where

$$a_\lambda\omega^{\lambda\rho}b_\rho = \sum_{\alpha,\beta}\{A,\Omega_\alpha\}\left(X\omega X^T\right)^{-1}_{\alpha\beta}\{\Omega_\beta,B\}\,.$$

From (7.11) we have, using matrix notation,

$$Q = -X\omega X^T\,.$$

Hence we obtain

$$\{\tilde{A},\tilde{B}\}_{\phi=0} = \{A,B\} - \sum_{\alpha,\beta}\{A,\Omega_\alpha\}Q^{-1}_{\alpha\beta}\{\Omega_\beta,B\} \equiv \{A,B\}_D\ .$$

This proves proposition 3. From (7.13) and the equations of motion for second class systems (see (3.69) with $v^{(1)}_{\alpha_1} = 0$) it follows that the latter are also given by

$$\dot{\tilde{A}}|_{\phi=0} = \{\tilde{A}, \tilde{H}\}_{\phi=0}$$
$$\tilde{\Omega}_\alpha|_{\phi=0} = 0.$$

Hence the requirements (7.4) to (7.9) are sufficient for ensuring that the correct (second-class) dynamics is reproduced within the gauge invariant sector of the embedded theory, as is made manifest above in the gauge $\phi^\alpha = 0$. [3]

7.3 The BFT construction

Let us now come to the actual procedure followed by BFT for constructing the tilde variables [Batalin 1991]. We will consider purely bosonic systems. The embedding procedure requires the construction of the first class constraints and first class Hamiltonian, satisfying (7.4) and (7.8), respectively. This is a purely algebraic problem whose solution is in general not unique. In fact, the only information regarding the new variables ϕ^α is that their Poisson brackets are given by (7.1) with $\omega^{\alpha\beta}$ satisfying (7.3). Assuming a locally regular embedding we now make the following power series ansatz for the first class constraints $\tilde{\Omega}_\alpha$ satisfying (7.4) and (7.5):

$$\tilde{\Omega}_\alpha = \Omega_\alpha + \sum_{n=1}^{\infty} \sum_{\alpha_1 \cdots \alpha_n} \Omega^{(n)}_{\alpha;\alpha_1\cdots\alpha_n}(q,p)\phi^{\alpha_1}\cdots\phi^{\alpha_n} \; . \tag{7.14}$$

Inserting this ansatz into (7.4) one finds that

$$\tilde{\Omega}_\alpha = \Omega_\alpha + \Omega^{(1)}_{\alpha;\beta}(q,p)\phi^\beta + \cdots \; , \tag{7.15}$$

where $\Omega^{(1)}_{\alpha;\beta}(q,p)$ must be a solution to

$$\left(\Omega^{(1)}_{\alpha;\rho}\Omega^{(1)}_{\beta;\sigma}\right)\omega^{\rho\sigma} = -Q_{\alpha\beta} \tag{7.16}$$

with $Q_{\alpha\beta}$ defined in (7.12). The solution to (7.16) is not unique. Thus, first of all, we have complete freedom in choosing an invertible antisymmetric matrix

[3] The dynamics of observables in the embedded theory in an arbitrary gauge, is given by $\dot{\tilde{A}} = \{\tilde{A}, \tilde{H}\}$. Note that it does not involve any Lagrange multipliers since observables are in strong involution with the first class constraints.

$\omega^{\alpha\beta}$; and secondly, having chosen $\omega^{\alpha\beta}$, the coefficients $\Omega^{(1)}_{\alpha;\beta}$ are only fixed up to the following transformations,

$$\Omega^{(1)}_{\alpha;\beta} \to \Omega^{(1)}_{\alpha;\gamma} S^{\gamma}{}_{\beta} \ ,$$

where

$$S^{\alpha}{}_{\gamma} S^{\beta}{}_{\delta} \omega^{\gamma\delta} = \omega^{\alpha\beta} \ .$$

Having obtained a solution for $\Omega^{(1)}_{\alpha;\beta}$, one next constructs the first class Hamiltonian $\tilde{H}(q,p,\phi)$ associated with the new set of constraints by making an ansatz analogous to (7.14), i.e.

$$\tilde{H} = H + \sum_{n=1}^{\infty} \sum_{\alpha_1 \cdots \alpha_n} H^{(n)}_{\alpha_1 \cdots \alpha_n}(q,p) \phi^{\alpha_1} \cdots \phi^{\alpha_n} \ ,$$

which satisfies the requirement (7.9). We remark that even if the new first class constraints are linear in the variables ϕ, the Hamiltonian $\tilde{H}$ will involve also bilinear expressions in the ϕ^{α}'s, since H is itself at least quadratic in the original variables (q,p). Here proposition 2 is of great help in the construction of the new Hamiltonian. This construction requires however the knowledge of the gauge invariant dynamical variables. In [Batalin 1987/91] prescriptions have been given for constructing the new Hamiltonian. [4] Rather than reproducing here the extensive set of formulae, we will illustrate the formalism in terms of a number of non-trivial examples. Before doing so it is however useful to discuss the basic ideas for a very simple quantum mechanical example involving only two degrees of freedom.

Consider the Lagrangian,

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x-y)^2 \ , \tag{7.17}$$

which, as we have seen in example 1 of chapter 3, describes a second class system with one primary and one secondary constraint:

$$\Omega_1 := p_y = 0 \quad \Omega_2 := p_x - 2y + x = 0 \ . \tag{7.18}$$

Since the constraints are second class, they merely fix the Lagrange multiplier v in the total Hamiltonian through the persistence of Ω_2: $v = \frac{1}{2}(p_x - x)$. The total Hamiltonian then takes the form,

$$H_T = \frac{1}{2}(p_x - y)^2 + \frac{1}{2}(x-y)^2 + \frac{1}{2}(p_x - x)p_y \ ,$$

[4] For an application of the prescription given in [Batalin 1987] see [Banerjee 1994].

and leads to the equations of motion

$$\begin{aligned} &\dot{x} = y - x \ , \quad \dot{y} = y - x \ , \\ &\dot{p}_x = y - x \ , \quad \dot{p}_y = p_x - 2y + x = 0 \ , \end{aligned} \tag{7.19}$$

together with the constraints (7.18). The rhs of these equations are nothing but the Dirac brackets of the dynamical variables with the canonical Hamiltonian. We now proceed to embed this system into a larger phase space by introducing two new degrees of freedom (corresponding to the number of constraints to be converted), which we choose to be a canonical pair (θ, p_θ). This corresponds to the choice $\omega^{\alpha\beta} = \epsilon^{\alpha\beta}$ in (7.1), and the identification $\theta = \phi^1$, $p_\theta = \phi^2$. By inspection we have for the new first class constraints

$$\tilde{\Omega}_1 := p_y + \theta = 0\,, \quad \tilde{\Omega}_2 := p_x + x - 2y - 2p_\theta = 0\,, \tag{7.20}$$

which are in strong involution: $\{\tilde{\Omega}_1, \tilde{\Omega}_2\} = 0$. From the requirements (7.6) and (7.7) for an observable we readily obtain for the tilde-variables,

$$\begin{aligned} &\tilde{x} = x + \frac{1}{2}\theta \ , \quad \tilde{y} = y + p_\theta \ , \\ &\tilde{p}_x = p_x - \frac{1}{2}\theta \ , \quad \tilde{p}_y = p_y + \theta \ , \end{aligned} \tag{7.21}$$

which are in strong involution with the first class constraints (7.20). These take the form

$$\tilde{\Omega}_1 := \tilde{p}_y = 0 \ , \quad \tilde{\Omega}_2 := \tilde{p}_x - 2\tilde{y} + \tilde{x} = 0 \tag{7.22}$$

and are seen to be of the same form as (7.18). The fundamental Poisson brackets of the tilde variables (7.21) are non-canonical:

$$\begin{aligned} &\{\tilde{x}, \tilde{y}\} = \frac{1}{2}, \quad \{\tilde{y}, \tilde{p}_x\} = \frac{1}{2} \ , \quad \{\tilde{x}, \tilde{p}_x\} = 1 \ , \quad \{\tilde{x}, \tilde{x}\} = 0 \ . \\ &\{\tilde{y}, \tilde{p}_y\} = 0 \ , \quad \{\tilde{x}, \tilde{p}_y\} = 0\,, \quad \{\tilde{p}_x, \tilde{p}_y\} = 0 \ . \end{aligned}$$

As one easily checks, these brackets (defined in the extended space) are identical with the corresponding Dirac brackets of the original second class system, in accordance with proposition 3. [5] Making use of proposition 2, a possible choice for the corresponding first class Hamiltonian is given by

$$\tilde{H} = H_0(\tilde{x}, \tilde{y}, \tilde{p}_x) \ ,$$

where

$$H_0(x, y, p_x) = \frac{1}{2}(p_x - y)^2 + \frac{1}{2}(x - y)^2$$

[5] Since the transformations (7.21) are linear in θ and p_θ, there is no need for setting $\theta = p_\theta = 0$.

is the canonical Hamiltonian evaluated on the primary surface $p_y = 0$ of the second class system. Hence

$$\tilde{H} = \frac{1}{2}(p_x - y - \frac{1}{2}\theta - p_\theta)^2 + \frac{1}{2}(x - y + \frac{1}{2}\theta - p_\theta)^2 \ . \tag{7.23}$$

Making use of (7.21) and the constraints (7.22), the equations of motion generated by $\tilde{H}$ according to $\dot{\tilde{A}} = \{\tilde{A}, \tilde{H}\}$ read

$$\dot{\tilde{x}} = \tilde{y} - \tilde{x}\,, \quad \dot{\tilde{y}} = \tilde{y} - \tilde{x}\,, \quad \dot{\tilde{p}}_x = \tilde{y} - \tilde{x}\,, \quad \dot{\tilde{p}}_y = 0\ ,$$

which are seen to be of the same form as (7.19). These must be supplemented by the constraints (7.22).

We now proceed to discuss some less simple examples.

7.4 Examples of BFT embedding

In the following we discuss a number of non-trivial models, both quantum mechanical and field theoretical, and their BFT-embedding into a gauge theory.

7.4.1 The multidimensional rotator

Consider the motion of a particle on a hypersphere S^{n-1}, as described by the Lagrangian [Hong 2004],

$$L = \frac{1}{2}\dot{\vec{q}}^{\,2} + \lambda \vec{q}\cdot\dot{\vec{q}}\,, \tag{7.24}$$

where $\{q_a\}$ $(a = 1, \dots, n)$ are the embedding coordinates for the S^{n-1} manifold, $\vec{q} = (q_1, \dots, q_n)$, and λ is the Lagrange multiplier implementing the second-class constraint $\vec{q}\cdot\dot{\vec{q}} = 0$, following from the geometrical constraint $\vec{q}\cdot\vec{q} = q^2 = constant$. [6] From (7.24) we obtain for the canonical momenta conjugate to the multiplier λ and the coordinates q_a: $p_\lambda = 0, \quad p_a = \dot{q}_a + \lambda q_a$. The canonical Hamiltonian on Γ_P reads

$$H_0(q, p, \lambda) = \frac{1}{2}\sum_{a=1}^{n}(p_a - \lambda q_a)^2. \tag{7.25}$$

The Dirac algorithm is readily shown to lead to a pair of second-class constraints,

$$\Omega_1 := p_\lambda = 0, \quad \Omega_2 := \vec{q}\cdot\vec{p} - \lambda q^2 = 0, \tag{7.26}$$

with the Poisson brackets

$$Q_{\alpha\beta} \equiv \{\Omega_\alpha, \Omega_\beta\} = \epsilon_{\alpha\beta} q^2\ .$$

[6] For an analogous discusssion of the $O(3)$ non-linear sigma model see [Hong 2003].

Here $q^2 = \sum_a q_a^2$. We now convert this algebra of second class constraints into a strongly involutive one, by suitably embedding the model into a larger dimensional phase space. Following the same procedure as before we introduce a pair of canonically conjugate variables (θ, p_θ). The new strongly involutive constraints can once again be constructed by inspection

$$\tilde{\Omega}_1 := \Omega_1 + \theta = 0 \ ; \ \tilde{\Omega}_2 := \Omega_2 - q^2 p_\theta = 0 \ . \tag{7.27}$$

Next we construct in the extended space the first-class variables $(\tilde{q}_a, \tilde{p}_a)$ as (in general non-canonical) gauge invariant extensions of the canonical pairs (q_a, p_a). They are obtained, as always, by making a power series ansatz in the auxiliary variables θ and p_θ and demanding that they be in strong involution with the first-class constraints. In the present example the answer cannot be obtained by simple inspection, but requires some work. After some tedious algebra, one finds [Hong 2004]

$$\begin{aligned}
\tilde{q}_a &= q_a \left(\frac{q^2 + 2\theta}{q^2}\right)^{1/2} , \\
\tilde{p}_a &= \left(p_a + 2q_a \lambda \frac{\theta}{q^2} + 2q_a p_\theta \frac{\theta}{q^2}\right)\left(\frac{q^2}{q^2 + 2\theta}\right)^{1/2} , \\
\tilde{\lambda} &= \lambda + p_\theta, \quad \tilde{p}_\lambda = p_\lambda + \theta \ .
\end{aligned} \tag{7.28}$$

Note that these solutions reflect an infinite power series in θ. One readily computes the following Poisson brackets in the embedding phase space,

$$\{\tilde{q}_a, \tilde{p}_b\} = \delta_{ab} \ , \quad \{\tilde{q}_a, \tilde{q}_b\} = 0 \ , \quad \{\tilde{p}_a, \tilde{p}_b\} = 0 \ , \tag{7.29}$$

which in this example happen to be canonical, as well as

$$\begin{aligned}
\{\tilde{\lambda}, \tilde{q}_a\} &= -\frac{\tilde{q}_a}{\tilde{q}^2} \ , \quad \{\tilde{\lambda}, \tilde{p}_a\} = \frac{\tilde{p}_a}{\tilde{q}^2} - 2\tilde{\lambda}\frac{\tilde{q}_a}{\tilde{q}^2} \ , \\
\{\tilde{p}_\lambda, \tilde{p}_a\} &= 0 \ , \quad \{\tilde{\lambda}, \tilde{p}_\lambda\} = 1 \ , \quad \{\tilde{p}_\lambda, \tilde{q}_a\} = 0 \ ,
\end{aligned}$$

which involve the gauge invariant extension of the Lagrange multiplier and its conjugate momentum. The rhs of these brackets evaluated at $\theta = p_\theta = 0$ are just the Dirac brackets of the corresponding variables in the second class system, with the constraints implemented strongly.

In terms of the coordinates (7.28) the Hamiltonian, in strong involution with the first class constraints, can be written - making use of proposition 2 - in the compact form

$$\tilde{H} = H_0(\tilde{q}, \tilde{p}, \tilde{\lambda}) = \frac{1}{2}\sum_a (\tilde{p}_a - \tilde{\lambda}\tilde{q}_a)^2 \ , \tag{7.30}$$

where $H_0(q,p,\lambda)$ is the canonical Hamiltonian (7.25). This is the explicit gauge invariant expression for the Hamiltonian which, written in terms of $(q_a, p_a, \lambda, \theta, p_\theta)$ reads,

$$\tilde{H}(q,p,\lambda,\theta,p_\theta) = \frac{1}{2}\eta^2 \sum_a (p_a - \lambda q_a - q_a p_\theta)^2 \,, \tag{7.31}$$

where

$$\eta = \left(\frac{q^2}{q^2 + 2\theta}\right)^{\frac{1}{2}} . \tag{7.32}$$

The strongly involutive constraints take the form

$$\tilde{\Omega}_1 := \tilde{p}_\lambda = 0, \quad \tilde{\Omega}_2 := \tilde{q}\cdot\tilde{p} - \tilde{\lambda}\tilde{q}\cdot\tilde{q} = 0,$$

which again display manifest form invariance with respect to the second-class constraints (7.26). The Hamiltonian (7.31) is in strong involution with these constraints.

Since $\tilde{\Omega}_i$ and all observables are now in strong involution with the constraints, we can impose the constraints strongly. Solving $\tilde{\Omega}_2 = 0$ for $\tilde{\lambda}$, one can thus effectively reduce the Hamiltonian (7.30) to the form

$$\tilde{H} = \frac{1}{2}\sum_a \left(\tilde{p}_a - \tilde{q}_a \frac{(\tilde{q}\cdot\tilde{p})}{\tilde{q}^2}\right)^2 . \tag{7.33}$$

Note that the $\tilde{q}_a$ and $\tilde{p}_a$ satisfy the canonical Poisson algebra (7.29). Hence in the first class formulation with the Hamiltonian (7.33) we can treat the particle motion as that of an unconstrained system.

7.4.2 The Abelian self-dual model

As a field-theory example we consider the self-dual (SD) model in 2+1 dimensions described by the Lagrangian density [Townsend 1984]

$$\mathcal{L}_{SD} = \frac{1}{2}f_\mu f^\mu - \frac{1}{2m}\epsilon_{\mu\nu\lambda} f^\mu \partial^\nu f^\lambda \,, \tag{7.34}$$

where $f^\mu = f^\mu(x)$. The corresponding field equations read

$$f_\mu - \frac{1}{m}\epsilon_{\mu\nu\lambda}\partial^\nu f^\lambda = 0 \,. \tag{7.35}$$

The Lagrangian (7.34) describes a purely second-class system with three primary constraints

$$\Omega_0 := \pi_0 = 0 \ , \quad \Omega_i := \pi_i + \frac{1}{2m}\epsilon_{ij} f^j = 0 \quad ; \quad (i,j = 1,2) \ , \tag{7.36}$$

and one secondary constraint

$$\Omega_3 := \frac{1}{m}(f^0 - \frac{1}{m}\epsilon_{ij}\partial^i f^j) = 0 \ . \tag{7.37}$$

Here π_μ are the canonical momenta conjugate to f^μ. The secondary constraint follows from the requirement of the persistence in time of $\Omega_0 = 0$:

$$\{\Omega_0(x), H_T\} \approx 0 \ ,$$

where H_T is the total Hamiltonian

$$H_T = H_0 + \sum_{\alpha=0}^{2} \int d^2x \ v^\alpha \Omega_\alpha \ ,$$

and the $v^\alpha(x)$'s are Lagrange multiplier fields. H_0 is the canonical Hamiltonian

$$H_0[f^0, f^i] = \int d^2x \ \left(-\frac{1}{2} f^\mu f_\mu + \frac{1}{m}\epsilon_{ij} f^0 \partial^i f^j \right) \ . \tag{7.38}$$

The non-vanishing (equal-time) Poisson brackets of the Ω_α's are given by

$$\begin{aligned}
\{\Omega_0(x), \Omega_3(y)\} &= -\frac{1}{m}\delta^2(x-y) \ , \\
\{\Omega_i(x), \Omega_j(y)\} &= \frac{1}{m}\epsilon_{ij}\delta^2(x-y) \ , \\
\{\Omega_i(x), \Omega_3(y)\} &= \frac{1}{m^2}\epsilon_{ij}\partial^j\delta^2(x-y) \ .
\end{aligned}$$

where $i,j = 1,2$. We now convert this second class system into a first class system by introducing a set of four scalar fields, $\phi^\alpha(x)$, $\alpha = 0,\dots,3$, corresponding to the number of constraints to be converted. Their Poisson algebra is given by (7.1), i.e.

$$\{\phi^\alpha(x), \phi^\beta(y)\} = \omega^{\alpha\beta}(x,y) \ .$$

Let us try the following simple linear ansatz for the new constraints, analogous to (7.15), [7]

$$\tilde{\Omega}_\alpha(x) = \Omega_\alpha(x) + \int d^2z \left[\int d^2y \ B_\alpha^{\ \gamma}(x,y)\omega_{\gamma\beta}(y,z) \right] \phi^\beta(z) \ , \tag{7.39}$$

[7] For the remaining part of this chapter it is understood that all space-time arguments x, y, z, etc. appearing in two-dimensional (spacial) integrals, such as e.g. (7.39), or as arguments in functions, such as $\omega_{\mu\nu}(x,y)$ and $B_\mu^{\ \nu}(x,y)$, are taken at equal times.

where $\omega_{\alpha\beta}$ is the inverse of $\omega^{\alpha\beta}$. As we have already emphasized, this antisymmetric matrix can be chosen freely. In terms of $B_\alpha^{\ \beta}$ relation (7.16) becomes

$$\int d^2\xi d^2\xi' \ B_\alpha^{\ \rho}(x,\xi)B_\beta^{\ \sigma}(y,\xi')\omega_{\sigma\rho}(\xi,\xi') = -Q_{\alpha\beta}(x,y) \ . \tag{7.40}$$

To obtain a solution for $B_\alpha^{\ \rho}(x,y)$, it is convenient to choose $\omega_{\alpha\beta}$ as follows [Banerjee 1997b]

$$\begin{aligned}
\omega_{03}(x,y) &= \frac{1}{m}\delta^2(x-y) \ , \\
\omega_{ij}(x,y) &= \frac{1}{m}\epsilon_{ij}\delta^2(x-y) \ , \\
\omega_{0i}(x,y) &= \frac{1}{m^2}\epsilon_{ik}\partial^k\delta^2(x-y) \ .
\end{aligned}$$

Having made a choice for $\omega_{\alpha\beta}$, the non-vanishing coefficient functions $B_\alpha^{\ \gamma}(x,y)$ in (7.39) are now determined from (7.40) to be

$$\begin{aligned}
B_0^{\ 3}(x,y) &= \delta^2(x-y) \ , \\
B_i^{\ j}(x,y) &= \delta^i_j\delta^2(x-y) \ ,
\end{aligned}$$

with $B_\beta^{\ \alpha}(x,y) = B_\alpha^{\ \beta}(y,x)$. The inverse matrix $\omega^{\alpha\beta}$ of $\omega_{\alpha\beta}$ is given by

$$\omega^{-1} = \begin{pmatrix} 0 & 0 & 0 & -m \\ 0 & 0 & -m & -\partial_1^x \\ 0 & m & 0 & -\partial_2^x \\ m & -\partial_1^x & -\partial_2^x & 0 \end{pmatrix} \delta^2(x-y) \, .$$

The first class functions $\tilde{\Omega}_\alpha$ are now readily obtained:

$$\begin{aligned}
\tilde{\Omega}_0(x) &= \Omega_0(x) - \frac{1}{m}\phi^0(x) \ , \\
\tilde{\Omega}_i(x) &= \Omega_i(x) + \frac{1}{m}\epsilon_{ik}\phi^k(x) + \frac{1}{m^2}\epsilon_{ik}\partial^k\phi^0(x) \ , \\
\tilde{\Omega}_3(x) &= \Omega_3(x) + \frac{1}{m}\phi^3(x) + \frac{1}{m^2}\epsilon_{k\ell}\partial^\ell\phi^k(x) \ .
\end{aligned} \tag{7.41}$$

One verifies that they are strongly involutive.

Having obtained the new constraints, we next construct the gauge invariant fields $\tilde{f}^\mu$, and then obtain the new gauge invariant Hamiltonian $\widetilde{H}$ according to proposition 2 from (7.38). For this we only need to know $\tilde{f}^0$ and $\tilde{f}^i$. These can be easily constructed by making again a linear ansatz in the auxiliary fields. One finds that

$$\tilde{f}^0 = f^0 + \phi^3 \, , \quad \tilde{f}^i = f^i + \phi^i \, .$$

Hence we can make the following choice for $\tilde{H}$:

$$\tilde{H} = H_0[f^0 + \phi^3, f^i + \phi^i] \ . \tag{7.42}$$

At the classical level we could of course work just as well with the second class formulation. Not so if we are interested in quantizing the system. In chapter 11 we will use the first class formulation to quantize this model.

7.4.3 Abelian self-dual model and Maxwell-Chern-Simons theory

It is instructive to discuss the above Abelian SD-model still from another point of view. In the previous example, *all* the constraints when converted to first class constraints satisfying a strongly involutive algebra. The corresponding embedded version was a gauge theory involving four extra auxiliary fields, and the connection between the original and the new constraints was fairly complicated.

Alternatively, one can also consider a partial embedding, in which only some of the constraints are implemented strongly via appropriate Dirac brackets, while the remaining ones are converted to first class making use of the BFT formalism. This leads again to a gauge theory but with fewer number of gauge degrees of freedom. Gauge invariant observables in this reduced gauge theory should have the same algebraic properties with respect to these Dirac brackets (which implement only a subsubset of constraints strongly), as their counterparts in the SD-model with all the constraints implemented strongly. The abelian SD-model is a simple example demonstrating this. At the same time this alternative approach allows one to prove - at least on classical level - its equivalence with the Maxwell-Chern-Simons theory. [8] The equivalence on quantum level will be demonstrated in chapter 11 using functional techniques.

In the following we choose to implement strongly the constraints Ω_i ($i = 1, 2$), with the (equal time) Poisson algebra [Banerjee 1995b]

$$\{\Omega_i(x), \Omega_j(y)\} \equiv Q_{ij}(x,y) = \frac{1}{m}\epsilon_{ij}\delta^2(x-y) \ .$$

This is accomplished by introducing the following Dirac bracket defined in the subset Ω_i of the constraints, and denoted in the following by D':

$$\begin{aligned} \{A(x), B(y)\}_{D'} &= \{A(x), B(y)\} \\ &- \sum_{i,j=1}^{2} \int d^2z d^2z' \{A(x), \Omega_i(z)\} Q^{-1}_{ij}(z,z')\{\Omega_j(z'), B(y)\} \ , \end{aligned} \tag{7.43}$$

[8] See [Deser 1984] for an alternative discussion.

where

$$Q_{ij}^{-1}(x,y) = -m\epsilon_{ij}\delta^2(x-y) \ .$$

Hence

$$\{A(x), \Omega_i(y)\}_{D'} \equiv 0 \ , \quad i = 1,2$$

for arbitrary $A(x)$. In particular we have $\{\Omega_i(x), \Omega_j(y)\}_{D'} \equiv 0$.

Below we list some relevant non-vanishing (equal time) Dirac brackets:

$$\begin{aligned}
\{f^i(x), f^j(y)\}_{D'} &= -m\epsilon_{ij}\delta^2(x-y) \ , \\
\{f^0(x), f^i(y)\}_{D'} &= 0 \ , \\
\{f^i(x), \pi_j(y)\}_{D'} &= \frac{1}{2}\delta_{ij}\delta^2(x-y) \ , \\
\{f^0(x), \Omega_0(y)\}_{D'} &= \delta^2(x-y) \ , \\
\{f^0(x), \Omega_3(y)\}_{D'} &= \{f^i(x), \Omega_0(y)\}_{D'} = 0 \ , \\
\{f^i(x), \Omega_3(y)\}_{D'} &= -\frac{1}{m}\partial^i\delta^2(x-y) \ , \\
\{\pi_i(x), \pi_j(y)\}_{D'} &= -\frac{1}{2}\epsilon_{ij}\delta^2(x-y) \ .
\end{aligned} \tag{7.44}$$

Consider next the Hamilton equations of motion. They can be written in the form

$$\dot{f}_\mu \approx \{f_\mu, H_0\}_{D'} + \sum_{\alpha=0,3} \lambda_\alpha \{f_\mu, \Omega_\alpha\}_{D'}$$

$$\dot{\pi}_\mu \approx \{\pi_\mu, H_0\}_{D'} + \sum_{\alpha=0,3} \lambda_\alpha \{\pi_\mu, \Omega_\alpha\}_{D'}$$

together with the constraints (7.36) and (7.37). [9] H_0 is the canonical Hamiltonian evaluated on the primary surface (7.38).

[9] Consider for simplicity a system with a finite number of degrees of freedom. The H-equations read

$$\dot{q}_i = \{q_i, H_0\} + \sum_{\alpha=0,3} \lambda_\alpha \{q_i, \Omega_\alpha\} + \sum_{r=1}^{2} \mu_r \{q_i, \Omega_r\} \ ,$$

$$\dot{p}_i = \{p_i, H_0\} + \sum_{\alpha=0,3} \lambda_\alpha \{p_i, \Omega_\alpha\} + \sum_{r=1}^{2} \mu_r \{p_i, \Omega_r\} \ ,$$

together with the constraints (7.36) and (7.37)

$$\Omega_r = 0, \quad \Omega_\alpha = 0 \ , \ r = 1,2 \ , \ \alpha = 0,3 \ .$$

From the persistence of the constraints $\Omega_r = 0$ one readily determines the Lagrange multi-

Note that Ω_0 and Ω_3 form a second class system with respect to the above Dirac brackets:

$$\{\Omega_3(x), \Omega_0(y)\}_{D'} = \frac{1}{m}\delta^2(x-y) \ . \tag{7.45}$$

We now embed this system into a gauge theory by converting the second class constraints $\Omega_0 = 0$ and $\Omega_3 = 0$ into a first class system with respect to the D'-Dirac brackets. Hence it is clear that in the following embedding procedure, all Poisson brackets in the BFT-presription discussed above must be replaced by the corresponding D'-Dirac brackets. Since only Ω_0 and Ω_3 are to be converted to first class constraints, we will require only two auxiliary fields. These can always be chosen as a canonical pair having vanishing D'-brackets with all other fields,

$$\{\alpha(x), \pi^{(\alpha)}(y)\}_{D'} = \delta^2(x-y) \ .$$

In the present case we can immediately write down the new first class constraints

$$\tilde{\Omega}_0 = \Omega_0 + \alpha \ , \tag{7.46}$$

$$\tilde{\Omega}_3 = \Omega_3 + \frac{1}{m}\pi^{(\alpha)} \ . \tag{7.47}$$

Making use of (7.45) we see that these are in strong involution,

$$\{\tilde{\Omega}_\alpha, \tilde{\Omega}_\beta\}_{D'} = 0 \quad , \quad \alpha, \beta = 0, 3 \ .$$

Note that the vanishing of these brackets does not follow from the definition of the Dirac bracket, but is a consequence of the embedding! In general $\{\mathcal{F}, \tilde{\Omega}_\alpha\}_{D'} \not\equiv 0$. In fact, the $\tilde{\Omega}_\alpha$'s ($\alpha = 0, 3$) are generators of infinitesimal gauge transformations:

$$\delta\mathcal{F}(x) = \sum_\alpha \int d^2y\ \theta_\alpha(y)\{\mathcal{F}, \tilde{\Omega}_\alpha\}_{D'} \ .$$

---

pliers μ_r:

$$\mu_r \approx -\sum_{r'} Q^{-1}_{rr'}\{\Omega_{r'}, H_\mathcal{T}\} \ ,$$

where $H_\mathcal{T} = H_0 + \sum_{\alpha=0,3} \lambda_\alpha \Omega_\alpha$. Inserting the expression for μ_r in the above expressions for $\dot{q}_i$ and $\dot{p}_i$ one finds that

$$\dot{q}_i \approx \{q_i, H_0\}_{D'} + \sum_{\alpha=0,3} \lambda_\alpha \{q_i, \Omega_\alpha\}_{D'} \ ,$$

$$\dot{p}_i \approx \{p_i, H_0\}_{D'} + \sum_{\alpha=0,3} \lambda_\alpha \{p_i, \Omega_\alpha\}_{D'} \ .$$

The Hamiltonian $\tilde{H}$ of the embedded theory should be invariant under these transformations in the strong sense. To obtain $\tilde{H}$ we make use of proposition 2, and first construct the gauge invariant extensions of $f^\mu(x)$. These can be immediately obtained by inspection. We denote them here by F^μ (rather than $\tilde{f}^\mu$):

$$\begin{aligned} F^0 &= f^0 + \pi^{(\alpha)} \ , \\ F^i &= f^i + \partial^i \alpha \ . \end{aligned} \tag{7.48}$$

Indeed, one verifies that

$$\{F^\mu, \tilde{\Omega}_\alpha\}_{D'} = 0 \ .$$

The gauge invariant Hamiltonian is then given by simply replacing f^μ in (7.38) by F^μ,

$$\tilde{H} = \int d^2x \left(\frac{1}{2} {F_0}^2 + \frac{1}{2} \vec{F}^2 - m F_0 \tilde{\Omega}_3 \right) \ , \tag{7.49}$$

where

$$\tilde{\Omega}_3 = \frac{1}{m} \left(F^0 - \frac{1}{m} \epsilon_{ij} \partial^i F^j \right)$$

is the embedded version of (7.37). The D'-brackets of the F_μ are given by

$$\begin{aligned} \{F^0(x), F^0(y)\}_{D'} &= 0 \ , \\ \{F^0(x), F^i(y)\}_{D'} &= \partial^i \delta^2(x-y) \ , \\ \{F^i(x), F^j(y)\}_{D'} &= -m \epsilon_{ij} \delta^2(x-y) \ . \end{aligned} \tag{7.50}$$

In the embedded theory the equations of motion for observables $\mathcal{O}$, satisfying

$$\{\mathcal{O}, \tilde{\Omega}_\alpha\}_{D'} = 0 \ ,$$

take the simple form

$$\dot{\mathcal{O}} = \{\mathcal{O}, \tilde{H}^{SD}\}_{D'} \ ,$$

where

$$H^{SD} = \frac{1}{2} \int d^2x \ (\vec{E}^2 + B^2) \, .$$

One then finds that the equations of motion for F^μ read

$$F_\mu - \frac{1}{m} \epsilon_{\mu\nu\lambda} \partial^\nu F^\lambda = 0 \ .$$

From here it follows that F^μ is divergenceless. Hence we can introduce fields A^μ as follows:

$$F_\mu = \epsilon_{\mu\nu\lambda} \partial^\nu A^\lambda \ .$$

Then (7.49) can be written in the form,

$$\tilde{H} = \frac{1}{2}\int d^2x \left(\vec{E}^2 + B^2\right) - \frac{1}{m}\int d^2x F_0 \left(\vec{\nabla}\cdot\vec{E} - mB\right) \tag{7.51}$$

where

$$B = -\epsilon_{ij}\partial^i A^j \tag{7.52}$$

has the form of a magnetic field in 2-space dimensions, and

$$E_i = \partial^i A^0 - \partial^0 A^i \tag{7.53}$$

is the corresponding electric field. The quantity within brackets in the last integral in (7.51) is just $m^2\tilde{\Omega}_3$,

$$m^2\tilde{\Omega}_3 = \vec{\nabla}\cdot\vec{E} - mB \ ,$$

and hence vanishes on the space of physical solutions.

The constraint structure and Hamiltonian resembles that of the Maxwell-Chern-Simons theory in 3 space-time dimensions, defined by the Lagrangian density

$$\mathcal{L}_{MCS} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m}{2}\epsilon_{\mu\nu\lambda}A^\mu\partial^\nu A^\lambda \ , \tag{7.54}$$

where the field strength's $F^{\mu\nu}$'s are related to the potentials A^μ in the usual way. The corresponding action is invariant under the transformations $A^\mu \to A^\mu + \partial^\mu\Lambda$. This model possesses one primary constraint: $\pi_0 = 0$. The canonical Hamiltonian evaluated on the primary surface has the form

$$H_0^{MCS} = \frac{1}{2}\int d^2x \ \left(\vec{E}^2 + B^2\right) - \int d^2x \ A^0 \left(\vec{\nabla}\cdot\vec{E} - mB\right) \tag{7.55}$$

where E_i and B are related to the potentials A^μ by (7.53) and (7.52). Written in terms of canonical variables, E_i is given by

$$E_i = -\pi_i + \frac{m}{2}\epsilon_{ij}A^j \ . \tag{7.56}$$

Persistence of the primary constraint $\pi_0 = 0$ yields the secondary constraint,

$$\Omega_1 := \vec{\nabla}\cdot\vec{E} - mB = 0$$

which formally coincides with $m^2\tilde{\Omega}_3 = 0$. Ω_1 and π_0 form a first class system. Making use of (7.52) and (7.56) one verifies that they are in strong involution with respect to standard Poisson brackets, and have vanishing Poisson brackets with the "electric" and "magnetic" fields (7.56) and (7.52). The first class constraints generate a local symmetry of the equations of motion. In the gauge invariant sector the dynamics is determined by the first integral on the rhs of (7.55),

$$\dot{\mathcal{O}} = \{\mathcal{O}, H^{MCS}\} \ ,$$

$$\pi_0 = 0 \ ; \quad \Omega_1 = 0 \ ,$$

where

$$H^{MCS} = \frac{1}{2}\int d^2x \ (\vec{E}^2 + B^2) \ .$$

The non-vanishing *Poisson* brackets of the E_i's and B are found to be

$$\{E_i(x), E_j(y)\} = -m\epsilon_{ij}\delta^2(x-y) \ ,$$
$$\{E_i(x), B(y)\} = \epsilon_{ij}\partial^j\delta^2(x-y) \ ,$$

and coincide with the corresponding *Dirac brackets* $\{,\}_{D'}$ of the embedded SD-model (7.50) (recall that in the SD-model $E_1 = F^2$, $E_2 = -F^1$ and $B_0 = -F^0$). Since the dynamics of gauge invariant quantities in the embedded SD-model is in turn equivalent to the second class dynamics of the corresponding quantities in the SD-model, we conclude that in the gauge invariant sector of the MCS theory this theory coincides with the SD-model.

7.4.4 The non-abelian SD model

As a less trivial application of the BFT formalism we consider the model of Townsend et al. [Townsend 1984] generalized to the non-abelian case, with Lagrangian density (7.34),

$$\mathcal{L} = -\frac{1}{2}trf_\mu f^\mu + \mathcal{L}_{CS} \ , \tag{7.57}$$

where $\mathcal{L}_{CS}$ is the Chern-Simons Lagrangian

$$\mathcal{L}_{CS} = \frac{1}{4m}\epsilon_{\mu\nu\rho}tr\left(F^{\mu\nu}f^\rho - \frac{2}{3}f^\mu f^\nu f^\rho\right).$$

Here f^μ are anti-hermitian Lie-algebra valued fields

$$f_\mu = t_a f^a_\mu \ ,$$

where $\{t_a\}$ are the generators of $SU(N)$ in the fundamental representation, and $F^{\mu\nu}$ is the usual matrix valued chromoelectric field strength tensor constructed from the f^μ's:

$$F^{\mu\nu} = \partial^\mu f^\nu - \partial^\nu f^\mu + [f^\mu, f^\nu].$$

Our conventions are

$$[t_a, t_b] = c_{ab}^c t_c \ ,$$
$$tr(t_a t_b) = -\delta_{ab} \ ,$$

where c_{ab}^c are the structure constants of the group.

In the case of the non-abelian self-dual model, the BFT embedding leads to an infinite power series in the auxiliary fields for the first-class phase space variables. We shall however see that the series can be summed, and that the auxiliary fields play the role of the Lie-algebra valued fields parametrizing a non-abelian gauge transformation [Kim 1998a].

We begin our BFT construction of the first class constraints with a notational remark. In the following, space-time indices like $\mu, \nu, \ldots, i, j \ldots$ will have covariant or contravariant character, as usual. The other (internal) indices $a, b \ldots$ will be placed either up or down in order to keep the expressions as transparent as possible.

Consider the non-abelian self-dual model (7.57). The momenta canonically conjugate to f_a^0 and f_a^i are given respectively by $\pi_0^a = 0$ and $\pi_i^a = -\frac{1}{2m}\epsilon_{ij} f_a^j$. We thus have two sets of primary constraints,

$$\Omega_0^a := \pi_0^a = 0, \tag{7.58}$$

$$\Omega_i^a := \pi_i^a + \frac{1}{2m}\epsilon_{ij} f_a^j = 0 \quad (i, j = 1, 2)\,. \tag{7.59}$$

These are the non-abelian analog of (7.36). The canonical Hamiltonian density associated with the above Lagrangian is given by

$$\mathcal{H}_0 = -\frac{1}{2} f_\mu^a f_a^\mu + \frac{1}{2m}\epsilon_{ij} f_0^a F_a^{ij}.$$

Persistence in time of the above constraints leads to one further (secondary) constraint, [10]

$$\Omega_3^a := f_0^a - \frac{1}{2m}\epsilon^{ij} F_{ij}^a = 0. \tag{7.60}$$

The constraints (7.58), (7.59) and (7.60) define a second-class system. In particular this is also true for the subset of constraints (7.59), as well as for (Ω_0^a, Ω_3^a). The Poisson-brackets of the former are given by

$$\{\Omega_i^a(x), \Omega_j^b(y)\} = \frac{1}{m}\epsilon_{ij}\delta^{ab}\delta^2(x-y) \ , \quad i = 1, 2 \ ,$$

[10] Note that the normalization of Ω_3^a differs from that in the abelian case.

while the latter form a canonical pair,

$$\{\Omega_3^a(x), \Omega_0^b(y)\} = \delta^{ab}\delta(x-y)\ .$$

Elements from different sets have vanishing Poisson brackets with each other. As in the case of the abelian self-dual model, we shall follow here the strategy of implementing the constraints $\Omega_i^a = 0$ strongly via Dirac brackets $\{\ ,\ \}_{D'}$ *defined in the subspace of these constraints.* [11] In that case, $\{\Omega_i^a(x), \Omega_j^b(y)\}_{D'} \equiv 0$, while for the remaining constraints $\Omega_0^a = 0$ and $\Omega_3^a = 0$ one finds the following Dirac brackets:

$$\{\Omega_\alpha^a(x), \Omega_\beta^b(y)\}_{D'} = Q_{\alpha\beta}^{ab}(x,y)\ ;\quad \alpha,\beta = 0,3 \tag{7.61}$$

with $Q_{\alpha\beta}^{ab}$ given in $\alpha - \beta$ space by the matrix [12]

$$Q_{\alpha\beta}^{ab}(x,y) := \begin{pmatrix} 0 & -\delta^{ab} \\ \delta^{ab} & -\frac{1}{2m}c_{ab}^d\epsilon_{kl}F_d^{kl} \end{pmatrix}\delta^2(x-y)\ . \tag{7.62}$$

We next reduce the second-class system with Dirac brackets (7.61) to a first-class system at the expense of introducing two sets of auxiliary fields $\{\phi_a^0\}$ and $\{\phi_a^3\}$ corresponding to $\{\Omega_0^a\}$ and $\{\Omega_3^a\}$, with Poisson (or D'-Dirac) brackets [13]

$$\left\{\phi_a^\alpha(x), \phi_b^\beta(y)\right\} = \omega_{ab}^{\alpha\beta}(x,y), \tag{7.63}$$

where we are free to make the choice

$$\omega_{ab}^{\alpha\beta}(x,y) = \epsilon^{\alpha\beta}\delta_{ab}\delta^2(x-y) \tag{7.64}$$

with $\epsilon^{03} = -\epsilon^{30} = 1$. The first-class constraints $\tilde\Omega_\alpha^a$ are now constructed as a power series in the auxiliary fields,

$$\tilde\Omega_\alpha^a = \Omega_\alpha^a + \sum_{n=1}^{\infty}\Omega_{\alpha;\alpha_1\cdots\alpha_n}^{(n)a;a_1\cdots a_n}\phi_{a_1}^{\alpha_1}\cdots\phi_{a_n}^{\alpha_n}\ , \tag{7.65}$$

where the expansion coefficent functions do not depend on the auxiliary fields, and are to be determined by the requirement that the constraints $\tilde\Omega_\alpha^a$ be strongly involutive *in the sense of the Dirac brackets* introduced above:

$$\left\{\tilde\Omega_\alpha^a(x), \tilde\Omega_\beta^b(y)\right\}_{D'} = 0. \tag{7.66}$$

[11] i.e., we define $\{A,B\}_{D'} \equiv \{A,B\} - \int d^2\xi d^2\xi'\ \{A,\Omega_i^a(\xi)\}(Q^{-1})_{ab}^{ij}(\xi,\xi')\{\Omega_j^b(\xi'),B\}$, where $Q_{ij}^{ab}(\xi,\xi') = \{\Omega_i^a(\xi), \Omega_j^b(\xi')\}$.

[12] The internal symmetry indices are placed at convenience.

[13] Actually the Poisson brackets coincide with the Dirac brackets, since the auxilary fields have vanishing Poisson brackets with Ω_i^a, $(i = 1,2)$.

Making the Ansatz

$$\tilde{\Omega}^a_\alpha(x) = \Omega^a_\alpha + \int d^2y X^{ab}_{\alpha\beta}(x,y)\phi^\beta_b(y) + \cdots \ ,$$

analogous to (7.10), and substituting this ansatz into (7.66) leads to lowest order in ϕ^β_b to the condition

$$\int d^2z d^2z' X^{ac}_{\alpha\lambda}(x,z)\omega^{\lambda\delta}_{cd}(z,z')X^{bd}_{\beta\delta}(z',y) = -Q^{ab}_{\alpha\beta}(x,y). \tag{7.67}$$

For the choice (7.64), this equation has (up to a natural arbitrariness) the solution [Kim 1998a]

$$X^{ab}_{\alpha\beta}(x,y) := \begin{pmatrix} \delta^{ab} & 0 \\ \frac{1}{4m}c^d_{ab}\epsilon_{kl}F^{kl}_d & \delta^{ab} \end{pmatrix} \delta^2(x-y). \tag{7.68}$$

In higher orders, the solution for $\tilde{\Omega}^a_\alpha$ can be obtained recursively and leads to the strongly involutive constraints [Kim 1998a, Banerjee 1997]

$$\tilde{\Omega}^a_0 = \pi^a_0 + \phi^0_a \tag{7.69}$$

$$\tilde{\Omega}^a_3 = f^a_0 - \frac{1}{2m}\epsilon_{ij}F^{ij}_a + \phi^3_a - \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!}\left[\left(\Phi^0\right)^n\right]_{ab}\left(\frac{1}{2m}\epsilon^{ij}F^b_{ij}\right) \tag{7.70}$$

where Φ^0 is the matrix valued field,

$$\Phi^0 = \sum_d \tau_d\phi^0_d \tag{7.71}$$

with τ^a in the adjoint representation:

$$\left(\Phi^0\right)_{ab} = c^d_{ab}\phi^0_d \ . \tag{7.72}$$

Let us write the above expression in a more transparent form. With the choice (7.64) for $\omega^{\alpha\beta}_{ab}$, we see that the fields ϕ^0_a and ϕ^3_a are conjugate to one another. Let us therefore define

$$\theta^a = \phi^0_a \ , \quad \pi^a_\theta = \phi^3_a \ . \tag{7.73}$$

Equations (7.69) and (7.70) then take the form

$$\tilde{\Omega}^a_0 = \pi^a_0 + \theta^a \ , \tag{7.74}$$

$$\tilde{\Omega}^a_3 = f^a_0 + \pi^a_\theta - V^a_b(\theta)(\frac{1}{2m}\epsilon_{ij}F^{ij}_b) \ , \tag{7.75}$$

where $V^a_{\ b}$ are the matrix elements of

$$V(\theta) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!}\theta^n \tag{7.76}$$

with

$$\theta = \tau_d \theta^d \quad ; \quad (\tau_d)_{ab} = c^d_{ab} \ .$$

The construction of the first-class Hamiltonian $\tilde{H}$ can in principle be carried out along similar lines as in the case of the constraints, by representing it as a power series in the auxiliary fields and requiring $\{\tilde{\Omega}^a_i, \tilde{H}\}_{D'} = 0$ subject to the condition $\tilde{H}[f, \phi = 0] = H_0$. It is however more economical to first construct the first class fields $\tilde{f}^\mu_a$. The new Hamiltonian is then given by $\tilde{H} = H_0[\tilde{f}]$. The "physical" fields $\tilde{f}^\mu_a$ are again obtained as a power series in the auxiliary fields ϕ^α_a by requiring them to satisfy

$$\{\tilde{\Omega}^a_\alpha, \tilde{f}^\mu_b\}_{D'} = 0 \ , \ \forall a, b \ .$$

The iterative solution of these equations leads to an infinite series which can be compactly written in terms of the Lie algebra valued field θ

$$\begin{aligned}
\tilde{f}^a_0 &= f^a_0 + \pi^a_\theta + (U^a_{\ b}(\theta) - V^a_{\ b}(\theta)) \left(\frac{1}{2m}\epsilon_{ij} F^b_{ij}\right) , \\
\tilde{f}^a_i &= U^a_{\ b}(\theta) f^b_i + V^a_{\ b}(\theta)\partial_i \theta^b \ ,
\end{aligned} \tag{7.77}$$

where $U(\theta)$ is given by

$$U(\theta) = e^\theta . \tag{7.78}$$

For the first class field tensor $\tilde{F}^{ij}_a$ one obtains,

$$\tilde{F}^a_{ij} = U^a_{\ b}(\theta) F^b_{ij}, \tag{7.79}$$

and for $\tilde{\pi}^a_0$ one finds

$$\tilde{\pi}^a_0 = \pi^a_0 + \theta^a . \tag{7.80}$$

We now observe that the first-class constraints $\tilde{\Omega}^a_0$ and $\tilde{\Omega}^a_3$ can be written in terms of the physical fields as follows

$$\begin{aligned}
\tilde{\Omega}^a_0 &= \tilde{\pi}^a_0 \\
\tilde{\Omega}^a_3 &= \tilde{f}^a_0 - \frac{1}{2m}\epsilon^{ij}\tilde{F}^a_{ij} .
\end{aligned}$$

Thus, as expected, the first class constraints are obtained from the second class constraints $\Omega^a_0 = 0$ and $\Omega^a_3 = 0$ (cf. eqs. (7.58) and (7.60)) by simply replacing f^a_μ and π^a_0 by their gauge invariant extensions. Having obtained the gauge invariant dynamical variables, we can write down immediately the first class Hamiltonian density again making use of proposition 2:

$$\tilde{\mathcal{H}} = -\frac{1}{2}\tilde{f}^a_\mu \tilde{f}^\mu_a + \frac{1}{2m}\epsilon_{ij}\tilde{f}^a_0 \tilde{F}^{ij}_a. \tag{7.81}$$

Let us interpret the above results. The fields $\tilde{f}^a_i$ have a simple interpretation. Defining the group valued field $g(\theta) = e^\theta$, where θ lies in the Lie-algebra of $SU(N)$, we have for any Lie algebra valued field $A = A^a t_a$ the relations

$$-tr(t_a g^{-1}(\theta) A g(\theta)) = U^a_{\ b}(\theta) A^b \ ,$$
$$-tr(t_a g^{-1}(\theta) \partial_\mu g(\theta)) = V^a_{\ b}(\theta) \partial_\mu \theta^b.$$

Making use of these relations we therefore see that the expression for $\tilde{f}_i = \tilde{f}^a_i t_a$ in (7.77) can be written in the compact form

$$\tilde{f}_i = g^{-1} f_i g + g^{-1}\partial_i g. \tag{7.82}$$

The fields $\tilde{f}^a_i$ are thus identified with the gauge-transform of the fields f^a_i. They are invariant under the extended gauge transformation

$$f^i \to h^{-1} f^i h + h^{-1}\partial^i h \ ,$$
$$g \to h^{-1} g \ ,$$

and thus are observables in the extended space. What concerns the first-class field strength tensor $\tilde{F}_{ij} = \tilde{F}^a_{ij} t_a$, it follows from (7.79) that

$$\tilde{F}_{ij} = g^{-1} F_{ij} g. \tag{7.83}$$

In [Kim 1998a] the above discussed embedded version of the non-abelian self-dual model was further explored by proving its equivalence to the theory defined by the Stückelberg Lagrangian. [14]

[14] For further examples of BFT embedding see e.g. [Banerjee 1993/95a/97a] and [Kim 1994/95/98b].

Chapter 8

Hamilton-Jacobi Theory of Constrained Systems

8.1 Introduction

The Hamilton-Jacobi (HJ) equation is a partial differential equation for the generating function of a canonical transformation from a set of canonical variables (q, p) to a new set (Q, P), whose dynamics is governed by a vanishing Hamiltonian $H_0'(Q, P, t) = 0$. This generating function, which - in standard textbook nomenclature - is of the type F_2 and denoted in the literature by $S(q, P, t)$, is a solution to

$$H_0' = \frac{\partial S}{\partial t} + H_0(q, \frac{\partial S}{\partial q}, t) = 0 \; , \tag{8.1}$$

where the old and new phase space variables are related implicitly by

$$p_i = \frac{\partial S}{\partial q^i} \; , \quad Q^i = \frac{\partial S}{\partial P_i} \; .$$

S is called the "Hamilton principle function" (HPF). Since the new Hamiltonian H_0' vanishes, the equations of motion for Q^i and P_i read $\dot{Q}^i = 0$, $\dot{P}_i = 0$, implying that

$$P_i(t) = \alpha_i \;\; ; \;\; Q^i(t) = \beta^i \; ,$$

where α_i and β^i are constants. In the following we denote the sets $\{\alpha_i\}$ and $\{\beta^i\}$ by α and β. As shown in any textbook on mechanics, these constants can be conveniently identified with the constants of integration inherent in the solution to the HJ-equation (8.1). Hence the solution $q^i(t)$ and $p_i(t)$ to the original Hamilton equations of motion are given implicitly by

$$p_i(t) = \frac{\partial S(q(t), \alpha, t)}{\partial q^i(t)} \ , \tag{8.2}$$

$$\beta^i = \frac{\partial S(q(t), \alpha, t)}{\partial \alpha_i} \ . \tag{8.3}$$

In textbooks on mechanics, the HJ-theory is usually discussed in connection with systems described by a non-singular Lagrangian. As we shall see, singular systems with only first class constraints can also be dealt with in a straightforward way. We shall consider them briefly in the following section. On the other hand, the HJ-formulation for second class systems is more subtle [Dominici 1984, Rothe 2003d].

The major part of this chapter is devoted to examples demonstrating various techniques which allow one to formulate the HJ-equations for second class systems. Our discussion will be restricted to systems with a time independent Hamiltonian, i.e. to conservative systems of the type studied in the previous chapters.

Let us first discuss in some detail the reason why the HJ-formulation for second class systems poses a problem. To this effect let us write (8.1) in the form

$$H_0'(\underline{q}, \underline{p}) = 0 \ , \tag{8.4}$$

where

$$H_0'(\underline{q}, \underline{p}) = p_0 + H_0(q, p) \ . \tag{8.5}$$

Here H_0 is the canonical Hamiltonian on Γ_P, and $\underline{q}$ and $\underline{p}$ stand for the collection $\{q^{\underline{i}}\}$ and $\{p_{\underline{i}}\}$, $\underline{i} = 0, 1, \cdots, n$ with [1]

$$q^0 = t \ , \ p_0 = \frac{\partial S}{\partial q^0} \ , \quad p_i = \frac{\partial S}{\partial q^i} \ . \tag{8.6}$$

Equation (8.4), together with (8.6), are the HJ equations for Hamilton's Principal Function S for the case of an *unconstrained* system.

8.1.1 Carathéodory's integrability conditions

Consider now a constrained system in which not all canonical variables are independent. As we have seen in chapters 2 and 3, the Lagrangian $L(q^i, \dot{q}^i)$ is singular in this case, i.e the determinant of the Hessian matrix $H_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$

[1] Actually, by assumption, H_0' does not depend on q^0.

vanishes. On the Hamiltonian level this implies the existence of primary constraints. Let m_1 be the number of such constraints. We write them in the form appropriate for our purposes [2]

$$H'_\alpha(\underline{q},\underline{p}) \equiv p_\alpha + H_\alpha(q,\{p_a\}) = 0\,, \quad (\alpha = 1,\cdots,m_1)\ , \tag{8.7}$$

where $\{p_i\} = (\{p_\alpha\},\{p_a\})$, $a = m_1+1,\cdots,n$, and where we have set $g_\alpha(q,\{p_a\}) = -H_\alpha$, with g_α defined in (3.5). Adjoining these constraints to (8.4), we are led to consider the coupled set of differential equations

$$H'_{\underline{\alpha}}\left(\underline{q},\frac{\partial S}{\partial \underline{q}}\right) \equiv \frac{\partial S}{\partial q^{\underline{\alpha}}} + H_{\underline{\alpha}}\left(q,\frac{\partial S}{\partial q^a}\right) = 0\ , \quad \underline{\alpha} = 0,\cdots,m_1\ . \tag{8.8}$$

Here we made use of the fact that, according to (3.8), H_0 does not depend on the p_α's. Continuing this line of reasoning, we supplement this system of differential equations with a similar set of equations associated with possible secondary constraints $H'_a(\underline{q},\underline{p}) = 0$. Let there be m_2 such constraints. We are then led to consider the following complete set of partial differential equations

$$H'_{\underline{\alpha}}(\underline{q},\frac{\partial S}{\partial \underline{q}}) = 0\ , \quad \underline{\alpha} = 0,1,\cdots,m_1\ , \tag{8.9}$$
$$H'_a(\underline{q},\frac{\partial S}{\partial \underline{q}}) = 0\ \ , \quad a = m_1+1,\cdots,m_1+m_2\ .$$

We denote this set by

$$H'_{\underline{A}}(\underline{q},\frac{\partial S}{\partial \underline{q}}) = 0, \quad \underline{A} = 0,1,\cdots,m_1+m_2\ . \tag{8.10}$$

This coupled set of differential equations only admit a solution provided the H'_A are in strong involution "on-shell", in the sense [Carathéodory 1967]

$$\{H'_{\underline{A}},H'_{\underline{B}}\}_{p_{\underline{i}}=\frac{\partial S}{\partial q^{\underline{i}}}} = 0\ , \tag{8.11}$$

where $H'_{\underline{A}}$ is considered as a function of $\underline{q}$ and $\underline{p}$, and S is a solution to (8.10). The (generalized) Poisson bracket is defined by ;

$$\{F,G\} = \sum_{\underline{i}=0}^{n}\left(\frac{\partial F}{\partial q^{\underline{i}}}\frac{\partial G}{\partial p_{\underline{i}}} - \frac{\partial G}{\partial q^{\underline{i}}}\frac{\partial F}{\partial p_{\underline{i}}}\right)\ . \tag{8.12}$$

In the following $\underline{i},\underline{j} = 0,1,\cdots,n$.

[2]We follow here the notation of [Güler 1992].

The above integrability conditions are easily derived from (8.10). We suppose that S has been found, and that eqs. (8.10) are thus satisfied identically. Then differentiating $H'_A(\underline{q}, \partial S/\partial \underline{q}) = 0$ and $H'_B(\underline{q}, \partial S/\partial \underline{q}) = 0$ with respect to $q^{\underline{i}}$, multiplying them by $-\frac{\partial H'_B}{\partial p_{\underline{i}}}$ and $\frac{\partial H'_A}{\partial p_{\underline{i}}}$, respectively, and adding them, we obtain

$$\{H'_A, H'_B\}_{\underline{p}=\frac{\partial S}{\partial \underline{q}}} = \sum_{\underline{i},\underline{j}=0}^{n} \frac{\partial^2 S}{\partial q^{\underline{i}} \partial q^{\underline{j}}} \left(\frac{\partial H'_A}{\partial p_{\underline{j}}} \frac{\partial H'_B}{\partial p_{\underline{i}}} - \frac{\partial H'_A}{\partial p_{\underline{i}}} \frac{\partial H'_B}{\partial p_{\underline{j}}} \right)_{\underline{p}=\frac{\partial S}{\partial \underline{q}}} . \tag{8.13}$$

The right hand side vanishes for symmetry reasons. Hence a function S satisfying (8.10) implies (8.11). Equations (8.11) are Carathéodory's integrability conditions, which ensure that for a first class system, the complete set of equations is integrable.

8.1.2 Characteristic curves of the HJ-equations

We next show that the integrability conditions imply the "on-shell" vanishing of the total differentials $dH'_{\underline{\alpha}}$ in the $2n+2$ dimensional phase space [Güler 1992]. [3]

Consider the canonical Hamiltonian on the primary surface. It is given by (3.7),

$$H_0 = p_a f^a + g_\alpha \dot{q}^\alpha - L(q, \dot{q}^\alpha, f^a) \ , \tag{8.14}$$

where we have made use of the primary constraints written in the form (3.5), and of (3.6). This Hamiltonian was shown in chapter 3 not to depend on the velocities $\dot{q}^\alpha$. Taking the derivative of (8.5) with respect to p_b, with H_0 given by (8.14), we obtain, after identifying $g_\alpha(q, \{p_a\})$ with $p_\alpha - H'_\alpha(\underline{q}, \underline{p})$,

$$\begin{aligned}\frac{\partial H'_0}{\partial p_b} &= f^b + p_a \frac{\partial f^a}{\partial p_b} - \frac{\partial H'_\alpha}{\partial p_b}\dot{q}^\alpha - \left(\frac{\partial L}{\partial \dot{q}^a}\right)_{\dot{q}^a = f^a} \frac{\partial f^a}{\partial p_b} \\ &= f^b - \frac{\partial H'_\alpha}{\partial p_b}\dot{q}^\alpha \ .\end{aligned}$$

Noting further from (8.6) that $\dot{q}^0 = 1$, we can write this in the form,

$$f^b - \sum_{\underline{\alpha}=0}^{n} \frac{\partial H'_{\underline{\alpha}}}{\partial p_b} \dot{q}^{\underline{\alpha}} = 0 \ .$$

[3] By "on-shell" we mean: "on the space of solutions to the HJ-equations".

From here - recalling that $\dot{q}^b = f^b(q, \{\dot{q}^\alpha\}, \{p_a\})$ - we are led to

$$dq^b = \frac{\partial H'_{\underline{\alpha}}}{\partial p_b} dq^{\underline{\alpha}} . \tag{8.15}$$

Adding the identities (following from (8.5) and (8.7)),

$$dq^{\underline{\beta}} = \frac{\partial H'_{\underline{\alpha}}}{\partial p_{\underline{\beta}}} dq^{\underline{\alpha}}$$

we obtain

$$dq^{\underline{i}} = \frac{\partial H'_{\underline{\alpha}}}{\partial p_{\underline{i}}} dq^{\underline{\alpha}} \quad , \quad \underline{i} = 0, 1, \cdots, n \ . \tag{8.16}$$

So far we have not made use of any dynamical equations. In fact, (8.15) divided by $dq^0 \equiv dt$, is nothing but the first set of Dirac's (kinematical) equations.

Setting $p_{\underline{i}} = \partial S/\partial q_{\underline{i}}$ we have

$$dp_{\underline{i}} = \frac{\partial^2 S}{\partial q^{\underline{\beta}} \partial q^{\underline{i}}} dq^{\underline{\beta}} + \frac{\partial^2 S}{\partial q^b \partial q^{\underline{i}}} dq^b \ .$$

By differentiating (8.8) with respect to q^i we obtain,

$$\frac{\partial^2 S}{\partial q^{\underline{i}} \partial q^{\underline{\alpha}}} + \left[\frac{\partial H'_{\underline{\alpha}}}{\partial q^{\underline{i}}} + \left(\frac{\partial H_{\underline{\alpha}}}{\partial p_a} \right) \frac{\partial^2 S}{\partial q^{\underline{i}} \partial q^a} \right]_{\underline{p} = \frac{\partial S}{\partial \underline{q}}} = 0 \ ,$$

where we made use of the fact that we could replace $H_{\underline{\beta}}$ by $H'_{\underline{\beta}}$ in the second term. Contracting this expression with $dq^{\underline{\alpha}}$ and making use of (8.16), we can cast $dp_{\underline{i}}$ into the form

$$dp_{\underline{i}} = -\frac{\partial H'_{\underline{\alpha}}}{\partial q^{\underline{i}}} dq^{\underline{\alpha}} \ . \tag{8.17}$$

These are nothing but the Hamilton equations of motion for the momenta generated by the total Hamiltonian of Dirac, with the momenta identified by $p_i = \partial S/\partial q^i$, and S a solution to the HJ-equations.

Consider further the differential $dS(\underline{q})$:

$$dS(\underline{q}) = \frac{\partial S}{\partial q^{\underline{\alpha}}} dq^{\underline{\alpha}} + \frac{\partial S}{\partial q^a} dq^a \ .$$

Making use of (8.8) and of (8.15), we can write it in the form

$$dS = \left(-H_{\underline{\beta}} + p_a \frac{\partial H'_{\underline{\beta}}}{\partial p_a} \right)_{p_{\underline{i}} = \frac{\partial S}{\partial q^{\underline{i}}}} dq^{\underline{\beta}} \ .$$

This equation, together with (8.16) and (8.17) are the total differentials for the characteristic curves of the HJ-equations. If they form a completely integrable set, their simultaneous solution determines S uniquely from the initial conditions. [4]

Let us examine what the integrability conditions (8.11) imply for the differential $dH'_{\underline{\beta}}(\underline{q}, \partial S/\partial q^a)$, with S a solution to the HJ-equations. Using (8.16) and (8.17) we have

$$\begin{aligned} dH'_{\underline{\beta}} &= \left(\frac{\partial H'_{\underline{\beta}}}{\partial q^{\underline{\alpha}}} + \frac{\partial H'_{\underline{\beta}}}{\partial q^a} \frac{\partial H'_{\underline{\alpha}}}{\partial p_a} - \frac{\partial H'_{\underline{\beta}}}{\partial p_a} \frac{\partial H'_{\underline{\alpha}}}{\partial q^a} \right) dq^{\underline{\alpha}} \\ &= \{H'_{\underline{\beta}}, H'_{\underline{\alpha}}\} dq^{\underline{\alpha}}, \end{aligned}$$

where $\{,\}$ is the generalized bracket defined in (8.12), or

$$dH'_{\underline{\beta}} = \{H'_{\underline{\beta}}, H'_{\underline{\alpha}}\} dq^{\underline{\alpha}}. \tag{8.18}$$

Thus the integrability condition (8.11) implies, in particular, the vanishing of dH'_0 and dH'_α, and hence Dirac's consistency equations, stating the time independence of the primary constraints.

On the other hand, for second class systems, these integrability conditions are evidently violated. One can attempt to circumvent this problem in several ways: i) One can look for a canonical change of variables in which the second-class constraints, or linear combinations thereof, become part of a new set of variables, which can be grouped into canonical pairs. If the Poisson brackets of the constraints are given by constant matrices, one can always perform such a transformation. This may however not be possible in general. ii) One can make use of the BFT procedure described in chapter 7 and embed the second class system into a first class one by suitably enlarging the phase space, Proceeding from there, we will present two ways in obtaining the solutions for the coordinates and conjugate momenta.

We begin our discussion with a purely first class system in order to exemplify that no integrability problem arises in this case.

8.2 HJ equations for first class systems

For first class systems the HJ equations (8.10) can be integrated without encountering any inconsistency. We illustrate this by the following example:

[4] The generalization to systems which include elements of the Berezin algebra has been carried out in [Pimentel 1998].

Consider the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x-y)^2 \ , \tag{8.19}$$

which leads to one primary and one secondary constraint,

$$H_1' := p_y = 0. \, , \quad H_2' := p_x - x = 0 \, , \tag{8.20}$$

respectively. The Hamiltonian is given by

$$H_0 = \frac{1}{2}p_x^2 - \frac{1}{2}x^2 - yp_x + xy$$

so that the integrability condition (8.11) is satisfied:

$$\{H_0', H_1'\} = H_2' \, , \quad \{H_0', H_2'\} = 0 \, , \quad \{H_1', H_2'\} = 0 \ .$$

The HJ-equation takes the form

$$H_0' = \frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial x} - y\right)^2 - \frac{1}{2}(x-y)^2 = 0 \, . \tag{8.21}$$

The constraint equations (8.20) lead to the additional conditions,

$$\frac{\partial S}{\partial y} = 0 \ , \quad \frac{\partial S}{\partial x} - x = 0 \ . \tag{8.22}$$

Let us rewrite (8.21) as follows:

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2 - \frac{1}{2}x^2 = \left(\frac{\partial S}{\partial x} - x\right) y \ . \tag{8.23}$$

Equations (8.22) together with (8.23) state that S does not depend on y or t. It then follows from $\frac{\partial S}{\partial x} - x = 0$ that

$$S = \frac{1}{2}x^2 \ .$$

Note that no integration constant (aside from an irrelevant additive constant) appeared! Hence, according to (8.2) and (8.3), we are left with only the equation $p_x = \partial S/\partial x$. This reflects the fact that the two Euler-Lagrange equations reduce to a single equation $\dot{x} - x = -y$, whose solution for $x(t)$ is only known once $y(t)$ is given. [5] Note that this equation is just the constraint equation

[5]For a given $y(t)$ the solution is given by

$$x = x(0)e^t - \left(\int_0^t d\tau \ e^{-\tau} y(\tau)\right) e^t \ .$$

$H'_2 = 0$. In fact, from (8.22) we recover the primary and secondary constraints,

$$p_y = \frac{\partial S}{\partial y} = 0\,, \quad p_x = \frac{\partial S}{\partial x} = x\ ,$$

and nothing more.

8.3 HJ equations for second class systems

Consider now a purely second class system where the integrability relation (8.11) is violated. Hence we do not expect to find a HPF from which one can obtain the solutions for the coordinates q and momenta p via (8.2) and (8.3).

In the following we will discuss two approaches for obtaining the HJ equations for such a system, of which the first one, to be discussed next, cannot be realized in general. The second approach makes use of the BFT embedding discussed in the preceding chapter, turning the system into a first class system, where gauge invariant quantities are identified with the corresponding quantities of the second class theory. The HJ-equations can then be integrated consistently. We illustrate both approaches with several examples.

8.3.1 HPF for reduced second class systems

Our strategy, to be illustrated by several examples, will be the following [Rothe 2003d]: given a Lagrangian we first generate all the constraints, following the Dirac algorithm. We then look for a possible canonical transformation from the set of phase space variables (q_i, p_i) to a new set consisting of two mutually "commuting" subsets [6] of canonically conjugate pairs, (q^*_a, p^*_a) and $(\tilde{q}_\alpha, \tilde{p}_\alpha)$, where $\{\tilde{q}_\alpha\}$ and $\{\tilde{p}_\alpha\}$ are identified with a suitably chosen complete set of primary and secondary constraints $\tilde{q}_\alpha = 0$ and $\tilde{p}_\alpha = 0$. Since the number of second class constraints is even, this can always be realized for linear constraints, but may be difficult, if not impossible, in the general case. Denoting by $F_2(q; p^*, \tilde{p}; t)$ the generating function for this canonical transformation, the new Hamiltonian will be given by

$$\tilde{H}(q^*, p^*, \tilde{q}, \tilde{p}) = H_0(q(q^*, p^*, \tilde{q}, \tilde{p}), p(q^*, p^*, \tilde{q}, \tilde{p})) + \frac{\partial F_2}{\partial t}\ , \qquad (8.24)$$

where $q(q^*, p^*, \tilde{q}, \tilde{p})$ and $p(q^*, p^*, \tilde{q}, \tilde{p})$ are obtained by solving the coupled set of equations

$$p_i = \frac{\partial F_2}{\partial q_i}\ , \quad \tilde{q}_\alpha = \frac{\partial F_2}{\partial \tilde{p}_\alpha}\ , \quad q^*_a = \frac{\partial F_2}{\partial p^*_a}\ . \qquad (8.25)$$

[6] "Commute" stands as a short hand for "having vanishing Poisson brackets".

In terms of the new variables the *extended* action reads, [7]

$$S_E = \int dt \left[\sum_a p_a^* \dot{q}_a^* + \sum_\alpha \tilde{p}_\alpha \dot{\tilde{q}}_\alpha - \tilde{H}(q^*, p^*, \tilde{q}, \tilde{p}) - \sum_\alpha \lambda_\alpha \tilde{q}_\alpha - \sum_\alpha \eta_\alpha \tilde{p}_\alpha \right] .$$

With the above canonical change of variables, the equations of motion for the "star" variables are determined from $\tilde{H}$ by implementing the constraints directly in the new Hamiltonian:

$$\dot{q}_a^* = \{q_a^*, \tilde{H}(q^*, p^*, 0, 0)\} , \quad \dot{p}_a^* = \{p_a^*, \tilde{H}(q^*, p^*, 0, 0)\} ,$$

while the equations for the remaining coordinates and momenta are just the persistency equations for the constraints. With respect to the phase-space coordinates (q^*, p^*) the problem has thus been reduced to that of an unconstrained system, and the Hamilton-Jacobi theory, described in the previous section, can be applied without encountering an inconsistency. Indeed, in terms of the new variables the relevant HJ-equations read

$$\frac{\partial S}{\partial t} + \tilde{H}(q^*, \frac{\partial S}{\partial q^*}, \tilde{q}, \frac{\partial S}{\partial \tilde{q}}) = 0$$

together with the constraint equations,

$$\tilde{q}_\alpha = 0 , \quad \frac{\partial S}{\partial \tilde{q}_\alpha} = 0 .$$

From these equations we see that S is only a function of q^* and t. Hence our problem reduces to solving the differential equations

$$H^*(q^*, \frac{\partial S}{\partial q^*}) + \frac{\partial S}{\partial t} = 0 ,$$

where

$$H^*(q^*, p^*) = \tilde{H}(q^*, p^*, \tilde{q} = 0, \tilde{p} = 0) .$$

The main problem therefore consists in finding the generating function F_2 which takes us from the set (q, p) to the new canonical sets (q^*, p^*) and $(\tilde{q}, \tilde{p})$.

[7] To formulate the general problem it is convenient to work with the extended action, where all of the constraints are implemented via Lagrange multipliers. Since all the constraints are generated from the total Hamiltonian H_T (involving only the primary constraints) via the Dirac self-consistent algorithm, the Lagrange multipliers associated with the *secondary* constraints will turn out to vanish weakly in a variational calculation (see eq. (3.63)).

8.3.2 Examples

In the following we will illustrate the above ideas in terms of some examples [Rothe 2003d]. We shall be rather detailed in the first example, in order to emphasize various aspects of the problem.

Example 1

Consider the Lagrangian (8.19) with a reversal of the sign in the last term:

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y - \frac{1}{2}(x-y)^2 \,. \tag{8.26}$$

The total Hamiltonian reads

$$H_T = H_0 + \eta p_y \,,$$

with

$$H_0 = \frac{1}{2}(p_x - y)^2 + \frac{1}{2}(x-y)^2$$

the canonical Hamiltonian evaluated on the primary surface $\phi :\equiv p_y = 0$. The Dirac algorithm leads to a secondary constraint $\varphi = 0$,

$$\{\phi, H_0\} = -2\varphi \;, \qquad \varphi = y - \frac{1}{2}(p_x + x) \,.$$

There are no further constraints. Since $\{\varphi, \phi\} = 1$, these constraints are second class and canonically conjugate to each other.

If we would proceed naively and make the replacement $p_y \to \frac{\partial S}{\partial y}$ in the primary constraint $p_y = 0$, then this would imply that S is independent of y. This would be in conflict with the equation obtained by making the substitution $p_x = \frac{\partial S}{\partial x}$ in the secondary constraint $\varphi = 0$. This reflects the non-commutative structure of our second class system, and corresponding violation of the integrability condition (8.11).

In order to circumvent this difficulty, we make a canonical transformation to a new set of variables (q^*, p^*) and $(\tilde{q}, \tilde{p})$, in which one of the canonically conjugate pairs are chosen to be the constraints themselves:

$$\begin{aligned} &\tilde{q} = \varphi \quad , \quad \tilde{p} = \phi \quad , \\ &q^* = x - \frac{1}{2}p_y \,, \quad p^* = p_x + \frac{1}{2}p_y \,. \end{aligned} \tag{8.27}$$

These are two "commuting" sets of canonically conjugate variables:

$$\{\tilde{q}, \tilde{p}\} = 1 \,, \quad \{q^*, p^*\} = 1 \,.$$

All other Poisson brackets vanish. As expected for a canonical transformation one also checks that

$$p_x\dot{x} + p_y\dot{y} = p^*\dot{q}^* + \tilde{p}\dot{\tilde{q}} + \frac{1}{2}\frac{d}{dt}(p^*\tilde{p} - \frac{1}{2}\tilde{p}^2) \ ,$$

so that the kinetic contribution to the action takes the same form in the new set of variables. The corresponding generating functional $F_2(x, y; p^*, \tilde{p})$ for the transformation (8.27) is readily constructed:

$$F_2 = (p^* - \frac{1}{2}\tilde{p})x + \tilde{p}y - \frac{1}{2}\tilde{p}p^* + \frac{1}{8}\tilde{p}^2 \, .$$

The inverse transformations to (8.27) read

$$\begin{aligned} x &= q^* + \frac{1}{2}\tilde{p}\,, \quad p_x = p^* - \frac{1}{2}\tilde{p}\,, \\ y &= \frac{1}{2}(q^* + p^*) + \tilde{q}\,, \quad p_y = \tilde{p}\,. \end{aligned} \tag{8.28}$$

Expressed in terms of the new variables the new total Hamiltonian $\tilde{H}_T$ takes the form

$$\tilde{H}_T = \tilde{H}_0 + \eta\tilde{p} \ ,$$

where

$$\tilde{H}_0(q^*, p^*, \tilde{q}, \tilde{p}) = \frac{1}{2}\left[\frac{1}{2}(p^* - q^*) - \left(\frac{\tilde{p}}{2} + \tilde{q}\right)\right]^2 + \frac{1}{2}\left[\frac{1}{2}(q^* - p^*) + \left(\frac{\tilde{p}}{2} - \tilde{q}\right)\right]^2 . \tag{8.29}$$

The equations of motions read,

$$\begin{aligned} \dot{q}^* &= \frac{1}{2}(p^* - q^*) - \frac{1}{2}\tilde{p}\,, \\ \dot{p}^* &= \frac{1}{2}(p^* - q^*) - \frac{1}{2}\tilde{p}\,, \\ \dot{\tilde{p}} &= -2\tilde{q}\,, \\ \dot{\tilde{q}} &= \eta - \frac{1}{2}(p^* - q^*) + \frac{1}{2}\tilde{p} \ , \end{aligned}$$

together with the constraints $\tilde{q} = \tilde{p} = 0$. From their persistency in time we see that the 3rd equation reproduces the secondary constraint $\tilde{q} = 0$, while the 4th equation fixes the Lagrange multiplier η as expected. The dynamics of the star variables on the constrained surface is now determined by the Hamiltonian $\tilde{H}_0(q^*, p^*, 0, 0) = \frac{1}{4}(p^* - q^*)^2$.

Consider now the HJ-equations associated with the constraints $\tilde{q} = \tilde{p} = 0$. With $\tilde{p}$ replaced by $\partial S/\partial\tilde{q}$ we conclude from $\tilde{p} = 0$ that the HPF does not depend on $\tilde{q}$. Hence the HJ-equation associated with $\tilde{H}_0$ becomes

$$\frac{\partial S}{\partial t} + \frac{1}{4}\left(\frac{\partial S}{\partial q^*} - q^*\right)^2 = 0 \ , \tag{8.30}$$

where we have set $\tilde{q} = 0$ and $\tilde{p} \to \partial S/\partial\tilde{q} = 0$ in (8.29).

Since the Hamiltonian $\tilde{H}$ is independent of time we can now make the usual separation ansatz

$$\frac{\partial S}{\partial t} = -\alpha^2 \ ,$$

with α a constant, so that equation (8.30) has the solution

$$S(q^*, \alpha, t) = \pm 2\alpha q^* + \frac{1}{2}q^{*2} - \alpha^2 t \ . \tag{8.31}$$

Notice that S depends only on one integration constant α. A further integration constant is then introduced according to (8.3),

$$\beta = \frac{\partial S}{\partial \alpha} \, ,$$

which then implies [8]

$$\beta = 2q^* - 2\alpha t \, .$$

With

$$p^* = \frac{\partial S}{\partial q^*} = \alpha t + 2\alpha + \frac{1}{2}\beta$$

the problem is now solved in the standard way. Taking account of the constraints $\tilde{q} = \tilde{p} = 0$ one obtains from (8.28)

$$x = \frac{1}{2}\beta + \alpha t \, , \quad y = (\alpha + \frac{1}{2}\beta) + \alpha t \ , \tag{8.32}$$

which is the most general solution to the Euler-Lagrange equations following from (8.26).

Example 2: *The Landau model in the zero mass limit*

The Lagrangian of a spinless charged particle, moving on a 2-dimensional plane in a harmonic oscillator potential in the presence of a constant background magnetic field B perpendicular to the 12-plane, can be written in the form

$$L_{Landau} = \frac{m}{2}\dot{\vec{q}}^{\,2} + \frac{B}{2}\vec{q} \times \dot{\vec{q}} - \frac{k}{2}\vec{q}^{\,2} \ ,$$

[8] Without loss of generality we have chosen the + sign in (8.31).

where (in two space dimensions) $\vec{q} \times \dot{\vec{q}} = \sum_{i,j=1}^{2} \epsilon_{ij} q^i \dot{q}^j$. In the zero-mass limit this Lagrangian reduces to

$$L = \frac{B}{2} \vec{q} \times \dot{\vec{q}} - \frac{k}{2} \vec{q}^2 \,.$$

There exist only two primary constraints

$$\phi_i = \frac{1}{\sqrt{B}} p_i + \frac{\sqrt{B}}{2} \epsilon_{ij} q^j$$

with non-vanishing Poisson brackets

$$\{\phi_i, \phi_j\} = \epsilon_{ij} \,.$$

We perform the following change of variables from q_i, p_i to q, p and $\tilde{p}_1 = \phi_1,\ \tilde{p}_2 = \phi_2$,

$$\begin{aligned} \tilde{p}_1 &= \frac{1}{\sqrt{B}} p_1 + \frac{\sqrt{B}}{2} q_2 \,, \quad \tilde{p}_2 = \frac{1}{\sqrt{B}} p_2 - \frac{\sqrt{B}}{2} q_1 \,, \\ q &= \frac{1}{2} q_1 + \frac{1}{B} p_2 \,, \quad p = p_1 - \frac{B}{2} q_2 \,, \end{aligned}$$

where q and p correspond to the "star variables" introduced before. The inverse transformations read

$$\begin{aligned} q_1 &= q - \frac{1}{\sqrt{B}} \tilde{p}_2 \,, \quad q_2 = \frac{1}{B} (\sqrt{B} \tilde{p}_1 - p) \,, \\ p_1 &= \frac{1}{2} (\sqrt{B} \tilde{p}_1 + p) \,, \quad p_2 = \frac{1}{2} (Bq + \sqrt{B} \tilde{p}_2) \,. \end{aligned} \tag{8.33}$$

In terms of the new coordinates the canonical Hamiltonian evaluated on the constrained surface takes the form,

$$H^* = \frac{k}{2} \left(q^2 + \frac{1}{B^2} p^2 \right) \,.$$

Hence we have for the HJ equation,

$$\frac{\partial S}{\partial t} + \frac{k}{2B^2} \left(B^2 q^2 + \left(\frac{\partial S}{\partial q} \right)^2 \right) = 0 \,,$$

with the solution

$$S(q, \alpha, t) = -\alpha t + B \int^q dq' \sqrt{\frac{2\alpha}{k} - q'^2} + const.$$

From the equation $\beta = \frac{\partial S}{\partial \alpha}$ we then obtain in the usual way the solution

$$q(t) = \sqrt{\frac{2\alpha}{k}} \cos(\omega t + b)\,,$$

with $\omega = k/B$ and b a constant.

We return to the original variables by setting in (8.33) the constraints $\tilde{p}_1$ and $\tilde{p}_2$ equal to zero, and using

$$p = \frac{\partial S}{\partial q} = B\sqrt{\frac{2\alpha}{k} - q^2}\,.$$

We thus obtain

$$q_1 = q\,, \quad q_2 = -\frac{p}{B} = -\sqrt{\frac{2\alpha}{k}} \sin(\omega t + b)\,.$$

8.3.3 HJ equations for second class systems via BFT embedding

As we have seen, a *naive* extension of the Hamilton-Jacobi theory to second class constrained systems is condemned to failure right from the outset since it leads to differential equations which are in direct conflict. Mathematically this conflict is expressed by the violation of the integrability condition (8.11). Above we have presented a possible way out of this dilemma. The drawback was that the transition to the reduced phase space is in practice restricted to rather simple situations.

We now present another method which is generally practicable. It is based on the replacement of the second class system by an equivalent first class system, using the BFT construction described in the previous chapter. The HJ equations are then integrable, since the system is in strong involution. Choose the auxiliary fields of the BFT construction to be sets of mutually commuting canonically conjugate pairs $(\theta^\alpha, p^\theta_\alpha)$, $\alpha = 1, \cdots, N$, where $2N$ is the number of 2nd class constraints. Since the number of second class constraints to be converted is even, this can be done, and implies a particular choice of the simplectic matrix $\omega^{\alpha\beta}$ in (7.1).

In the following we denote by $\tilde{S}(q, \theta, \alpha, t)$ the HPF of the BFT embedded system, with α a set of integration constants, and by $\tilde{\Omega}_A(q, \theta; p, p^\theta) = 0$ the set of first class constraints in strong involution. If $\tilde{H}(q, \theta, p, p_\theta)$ is the Hamiltonian, then $\tilde{S}$ is a solution to the following set of equations

$$\frac{\partial \tilde{S}}{\partial t} + \tilde{H}\left(q, \theta, \frac{\partial \tilde{S}}{\partial q}, \frac{\partial \tilde{S}}{\partial \theta}\right) = 0, \tag{8.34}$$

$$\tilde{\Omega}_A\left(q, \theta, \frac{\partial \tilde{S}}{\partial q}, \frac{\partial \tilde{S}}{\partial \theta}\right) = 0 \ . \tag{8.35}$$

From $\tilde{S}(q, \theta, \alpha, t)$ we then obtain the solutions for the embedding coordinates q_i, θ^α, and conjugate momenta p_i, p_α by expressions analogous to (8.2) and (8.3):

$$p_i(t) = \frac{\partial \tilde{S}}{\partial q_i(t)} \ , \ \beta_\ell = \frac{\partial \tilde{S}}{\partial \alpha_\ell} \ , \tag{8.36}$$

$$p^\theta_\alpha(t) = \frac{\partial \tilde{S}}{\partial \theta^\alpha(t)} \ . \tag{8.37}$$

Having obtained a solution $\tilde{S}$ of the HJ-equations for the embedded system we now want to arrive at the dynamical solution for the coordinates and momenta of the original second class theory. Before we discuss this in detail we want to make several assertions [Kleinmann 2004].

Assertion

Let $\Omega_A(q,p) = 0$ *be the constraints of a second class theory, and* $\tilde{\Omega}_A(q, \theta; p, p_\theta) = 0$ *the corresponding first class constraints of its embedded version. In accordance with the BFT power series expansions of* $\tilde{\Omega}_A$ *and* $\tilde{H}$*, assume the embedding to be analytic in* θ and p_θ. *Denote by* $\tilde{S}(q, \theta, \alpha, t)$ *the HPF of the embedded theory, satisfying (8.34) and (8.35), where* α *stands for a set of integration constants. Furthermore define* $S(q, \alpha, t) \equiv \tilde{S}(q, \theta = 0, \alpha, t)$. *We then claim that*:

$S(q, \alpha, t)$ solves (8.8) only in the "weak sense",

$$\frac{\partial S}{\partial t} + H_0\left(q, \frac{\partial S}{\partial q}\right) = \mathcal{P}(s) \ , \tag{8.38}$$

$$\Omega_A\left(q, \frac{\partial S}{\partial q}\right) = \mathcal{P}_A(s) \tag{8.39}$$

where s stands for $\{s_A\}$ with

$$s_A = (\partial \tilde{S}/\partial \theta^A)_{\theta=0} \ ,$$

and where $\mathcal{P}(s)$ and $\mathcal{P}_A(s)$ are polynomials in s satisfying $\mathcal{P}(0) = \mathcal{P}_A(0) = 0$. Note that these polynomials cannot all vanish identically, since this would be in conflict with Carathéodory's theorem.

The proof is immediate: Consider the HJ-equation (8.34). Setting $\theta = 0$ we have that

$$\frac{\partial S}{\partial t} + \tilde{H}_0\left(q, \theta = 0, \frac{\partial S}{\partial q}, s\right) = 0 \ .$$

Because of the assumed analyticity of the embedding in s, it follows that

$$\tilde{H}_0(q,\theta=0,\frac{\partial S}{\partial q},s) = H_0(q,\frac{\partial S}{\partial q}) + \mathcal{P}(s) \ .$$

A similar reasoning applies to the constraint equation (8.35). We are therefore led to (8.39).

How to get the solution to the second class system

i) A solution for the phase space variables of the second class theory is obtained by a) solving the complete set of HJ equations for the HPF of the embedded first class theory, b) determining the time dependence of these variables from (8.36) and (8.37), and c) imposing the conditions

$$\theta^\alpha = 0 \ , \quad \left(\frac{\partial \tilde{S}}{\partial \theta^\alpha}\right)_{\theta=0} = 0 \ , \forall \alpha \ . \tag{8.40}$$

This amounts to the choice of gauge $\theta^\alpha = 0$, $p^\theta_\alpha = 0$, which takes us back to the second class theory.

ii) A solution for the phase space variables of the second class theory is obtained by a) computing from $S(q,\alpha,t) \equiv \tilde{S}(q,\theta=0,\alpha,t)$ the phase space variables as a function of time from (8.2) and (8.3), and b) taking explicit account of the second class constraints $\Omega_A = 0$ in the form

$$\Omega_A\left(q,\frac{\partial S}{\partial q}\right) = 0 \ , \quad \forall A \ . \tag{8.41}$$

This statement follows from *i)*. Thus, by setting $\theta^\alpha = 0$ in (8.36), one is led to the equations $p_i = \partial S/\partial q^i$ and $\beta_\ell = \partial S/\partial \alpha_\ell$. These equations are just those for an unconstrained system with HPF $S(q,\beta,t)$, i.e., (8.2) and (8.3). The remaining equations $p^\theta_\alpha = (\partial \tilde{S}/\partial \theta^\alpha)_{\theta=0} = 0$ implement (8.41), as seen from (8.39) by setting $s = 0$. [9]

iii) Compute the solution in the embedded formulation for the gauge invariant combinations $\tilde{q}$ and $\tilde{p}$, and make use of the fact that these coincide with the solutions in the second class theory.

[9] Note that "constraint equations" (8.41), arising from (8.35) where $\tilde{\Omega}_A$ does not depend on p^θ_α, are automatically satisfied.

8.3.4 Examples

Let us verify the above claims in some examples.

Example 1

Consider the embedded formulation of the model described by the Lagrangian (7.17). Our starting point is therefore the embedded Hamiltonian (7.23), i.e.

$$\tilde{H} = \frac{1}{2}(p_x - y - \frac{\theta}{2} - p_\theta)^2 + \frac{1}{2}(x - y + \frac{\theta}{2} - p_\theta)^2 .$$

From this Hamiltonian and the first class constraints, $\tilde{\Omega}_1 = p_y + \theta = 0\,, \tilde{\Omega}_2 = p_x + x - 2y - 2p_\theta = 0$, we obtain for the complete set of HJ equations (8.10)

$$\frac{\partial \tilde{S}}{\partial t} + \frac{1}{2}\left(\frac{\partial \tilde{S}}{\partial x} - y - \frac{1}{2}\theta - \frac{\partial \tilde{S}}{\partial \theta}\right)^2 + \frac{1}{2}\left(x - y + \frac{1}{2}\theta - \frac{\partial \tilde{S}}{\partial \theta}\right)^2 = 0\ , \tag{8.42}$$

$$\frac{\partial \tilde{S}}{\partial y} + \theta = 0\ ,\quad \frac{\partial \tilde{S}}{\partial x} + x - 2y - 2\frac{\partial \tilde{S}}{\partial \theta} = 0\,. \tag{8.43}$$

As before we make the separation ansatz $\frac{\partial \tilde{S}}{\partial t} = -\alpha$. The solution to (8.42) is then given by having each term in brackets to be a constant:

$$\frac{\partial \tilde{S}}{\partial x} - y - \frac{1}{2}\theta - \frac{\partial \tilde{S}}{\partial \theta} = \gamma_1\ ,$$
$$x - y + \frac{1}{2}\theta - \frac{\partial \tilde{S}}{\partial \theta} = \gamma_2\ .$$

The constants α, γ_1 and γ_2 are constrained to satisfy

$$\alpha = \frac{1}{2}(\gamma_1^2 + \gamma_2^2)\,.$$

Joining these equations to the two constraint equations (8.43), we conclude that $\gamma_1 + \gamma_2 = 0$, so that

$$\alpha = \gamma_1^2 = \gamma_2^2\,.$$

Integrating sequentially the above differential equations for $\tilde{S}$ one finds,

$$\tilde{S} = (x-y)\theta + \frac{1}{4}\theta^2 + \sqrt{\alpha}\theta + \frac{1}{2}x^2 + 2\sqrt{\alpha}x - \alpha t + const\,. \tag{8.44}$$

From here we see that

$$\left(\frac{\partial \tilde{S}}{\partial \theta}\right)_{\theta=0} = x - y + \sqrt{\alpha}\ ,$$

which is seen *not* to vanish. In fact, in agreement with our above assertion (8.39), the rhs is proportional to the second class constraint $\Omega_2 = p_x + x - 2y$ with p_x replaced by $\partial S/\partial x$,

$$2\Omega_2\left(x, y, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}\right) = \left(\frac{\partial \tilde{S}}{\partial \theta}\right)_{\theta=0} ,$$

where

$$S = \tilde{S}|_{\theta=0} = \frac{1}{2}x^2 + 2\sqrt{\alpha}x - \alpha t + const \ . \tag{8.45}$$

Furthermore, with S given by (8.45),

$$\frac{\partial S}{\partial t} + H_0\left(q, \frac{\partial S}{\partial q}\right) = \frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial x} - y\right)^2 + \frac{1}{2}(x-y)^2 = \left(\frac{\partial \tilde{S}}{\partial \theta}\right)^2_{\theta=0} ,$$

in accordance with our assertion (8.38). Notice that constraints $\Omega_\alpha = 0$ which do not involve p^θ_α, such as $\Omega_1 = 0$ in this example are automatically satisfied by our solution.

Finally, starting from the HPF for the embedded model, let us recover the solutions of the Euler-Lagrange equations of motion for the second class system. Following the procedure described in *i)* we compute from (8.44), and the equations (8.3) and (8.2) the coordinates and momenta as a function of time in the embedded theory:

$$\begin{aligned}
p_x &= x + 2\sqrt{\alpha} + \theta \ , \\
p_y &= -\theta \ , \\
p^\theta &= x - y + \frac{1}{2}\theta + \sqrt{\alpha} \ , \\
\beta &= \frac{1}{\sqrt{\alpha}}x - t + \theta \ .
\end{aligned}$$

Imposing the gauge conditions (8.40), leads to the solutions of the original second class system, and in particular to

$$\begin{aligned}
x &= \sqrt{\alpha}t + \sqrt{\alpha}\beta \\
y &= x + \sqrt{\alpha} = \sqrt{\alpha}t + \sqrt{\alpha}(\beta + 1) \ .
\end{aligned} \tag{8.46}$$

The same exercise can be repeated following the prescription given in assertion *ii)*. Thus from (8.45) and the equations (8.36) and (8.37) we obtain that $p_x(t) = x(t) + 2\sqrt{\alpha}$, $p_y(t) = 0$, while from (the constraint equations) (8.41) we determine $y(t)$: $y(t) = x(t) + \sqrt{\alpha}$.

Example 2

Consider the Lagrangian [Kleinmann 2004]

$$L = \frac{1}{2}\dot{x}y^2 - y \ ,$$

with the canonical Hamiltonian $H_0 = y$ and constraints

$$\Omega_1 := p_x - \frac{1}{2}y^2 = 0 \ , \quad \Omega_2 := p_y = 0 \ . \tag{8.47}$$

The embedded Hamiltonian and associated first class constraints read

$$\begin{aligned}\tilde{H} &= y + p_\theta \ , \\ \tilde{\Omega}_1 &= p_x - \frac{1}{2}(y + p_\theta)^2 = 0 \ , \\ \tilde{\Omega}_2 &= p_y + \theta = 0 \ .\end{aligned} \tag{8.48}$$

This Hamiltonian and the constraints are in strong involution by construction. The HPF is obtained as the solution of the partial differential equations

$$\frac{\partial \tilde{S}}{\partial t} + y + \frac{\partial \tilde{S}}{\partial \theta} = 0 \ ,$$

$$\frac{\partial \tilde{S}}{\partial x} - \frac{1}{2}(y + \frac{\partial \tilde{S}}{\partial \theta})^2 = 0 \ ,$$

$$\frac{\partial \tilde{S}}{\partial y} + \theta = 0 \ .$$

With $\frac{\partial \tilde{S}}{\partial t} = -\alpha$ this system of equations reduces to

$$\begin{aligned}\frac{\partial \tilde{S}}{\partial \theta} &= \alpha - y \ , \\ \frac{\partial \tilde{S}}{\partial x} &= \frac{1}{2}\alpha^2 \ , \\ \frac{\partial \tilde{S}}{\partial y} &= -\theta \ .\end{aligned}$$

Hence the HPF of the embedded system is given by

$$\tilde{S} = -\alpha t + \frac{1}{2}\alpha^2 x - y\theta + \alpha\theta \ . \tag{8.49}$$

Defining

$$S = \tilde{S}|_{\theta=0} = -\alpha t + \frac{1}{2}\alpha^2 x \ , \tag{8.50}$$

we see that

$$\frac{\partial S}{\partial t} + H_0\left(q, \frac{\partial S}{\partial q}\right) = -\left(\frac{\partial \tilde{S}}{\partial \theta}\right)_{\theta=0} \ ,$$

where $q = (x, y)$ and $H_0 = y$. The lhs is the expression appearing in the naive HJ-equation for the original second class system, while the rhs is given by

$$\left(\frac{\partial \tilde{S}}{\partial \theta}\right)_{\theta=0} = -\frac{2}{\alpha + y}\Omega_1\left(x, y, \left(\frac{\partial S}{\partial x}\right), \left(\frac{\partial S}{\partial y}\right)\right) \ .$$

This is in agreement with the above made assertion. Notice that $\Omega_2 = 0$ is satisfied identically. Following the prescription given *i)*, we obtain the solution for the coordinates and momenta of the second class theory by first computing the corresponding coordinates and momenta from (8.49), (8.36) and (8.37). This yields

$$\beta = \frac{\partial \tilde{S}}{\partial \alpha} = -t + \alpha x + \theta \ .$$

Imposing the gauge condition (8.40) on (8.49) leads to

$$x = \frac{1}{\alpha}(t + \beta) \ , \quad y = \alpha \ .$$

Alternatively, following the prescription given in assertion *ii)*, the solution for x, y, p_x and p_y of the second class theory is obtained by computing the coordinates and momenta according to (8.2) and (8.3), with S given by (8.50), and implementing the constraints (8.41). This leads again to the above exprssions.

Example 3

Let us consider once more the multidimensional rotator (7.24). The HJ equations for the embedded system, described by the Hamiltonian (7.31) and the first-class constraints (7.27), read

$$\frac{\partial \tilde{S}}{\partial t} + \frac{1}{2}\eta^2 \sum_a \left(\frac{\partial \tilde{S}}{\partial q_a} - q_a\frac{\partial \tilde{S}}{\partial \theta} - \lambda q_a\right)^2 = 0 \, , \tag{8.51}$$

$$\frac{\partial \tilde{S}}{\partial \lambda} + \theta = 0 \, , \tag{8.52}$$

$$\sum_a q_a \frac{\partial \tilde{S}}{\partial q_a} - q^2\frac{\partial \tilde{S}}{\partial \theta} - \lambda q^2 = 0 \, , \tag{8.53}$$

where $a = 1, \cdots, N$. Consider the constraint equation (8.52). With the standard separation ansatz it has the solution

$$\tilde{S}(t, q_a, \lambda, \theta) = -\frac{\alpha^2}{2}t + \tilde{W}(q_a, \theta) - \lambda\theta \ ,$$

where $\tilde{W}(q, \theta)$ is so far an undetermined function. Hence (8.51) and (8.53) become respectively

$$-\frac{\alpha^2}{2} + \frac{1}{2}\eta^2 \sum_a \left(\frac{\partial \tilde{W}}{\partial q_a} - q_a \frac{\partial \tilde{W}}{\partial \theta} \right)^2 = 0. \tag{8.54}$$

$$q_a \frac{\partial \tilde{W}}{\partial q_a} - q^2 \frac{\partial \tilde{W}}{\partial \theta} = 0 \, . \tag{8.55}$$

A trivial solution of equation (8.55) is

$$\tilde{W}(q, \theta) = g(q^2 + 2\theta) = g(\tilde{q}^2) \ ,$$

which, according to (8.54), implies however the vanishing of α. Here $\tilde{q}_a$ are the first class variables defined in (7.28). Let us therefore look for a non-trivial solution with $\alpha \neq 0$. Instead of the above trivial solution we make the ansatz

$$\tilde{W}(q, \theta) = f(n \cdot \tilde{q})$$

where n_a are the components of an n-dimensional unit vector parametrized by $n - 1$ constants. Using

$$\frac{\partial \tilde{q}_c}{\partial q_a} = \eta^{-1}\delta_{ca} - \eta \frac{2\theta}{q^2} \frac{q_a q_c}{q^2} \, , \qquad \frac{\partial \tilde{q}_c}{\partial \theta} = \eta \frac{q_c}{q^2} \, , \tag{8.56}$$

with η defined in (7.32), one readily checks that equation (8.55) is satisfied for any $f(x)$. This function is determined from (8.54), which now reads,

$$\left(1 - \frac{(n \cdot \tilde{q})^2}{\tilde{q} \cdot \tilde{q}} \right) f'^2(n \cdot \tilde{q}) = \alpha^2 \ ,$$

where the prime denotes the derivative with respect to the argument of f. The solution to this equation has been given in [Rothe 2003d]. Setting $x = n \cdot \tilde{q}$ and $r^2 = \tilde{q}^2$, we have

$$f'(x) = \pm \frac{\alpha}{\sqrt{1 - \frac{x^2}{r^2}}} \ ,$$

so that, upon integration in x, the Hamilton principal function $\tilde{W}$ takes the form

$$\tilde{W}(q, \theta) = \alpha r \ \tan^{-1} \frac{n \cdot \tilde{q}}{\sqrt{r^2 - (n \cdot \tilde{q})^2}} + const \, . \tag{8.57}$$

$\tilde{W}$ involves N parameters, which we choose to be α and $n_1, \cdots, n_{N-1}$, while the normalization of n^a implies for the nth component, $n_n = \sqrt{1 - \sum_{a=1}^{N-1} n_a n_a}$. Differentiating the Hamilton principal function with respect to these constants (new momenta in the corresponding generating functional) yields, according to (8.2) and (8.3) the N time-independent new coordinates:

$$\beta = \frac{\partial \tilde{W}}{\partial \alpha}\,, \quad \beta_a = \frac{\partial \tilde{W}}{\partial n_a}\,, \quad a = 1, 2, \ldots, N-1\,.$$

From the first equation and (8.57) we easily obtain

$$n \cdot \tilde{q} = r \cos \frac{\beta + \alpha t}{r} \equiv r \cos \Omega(t)\,.$$

The solution for the "observable" $\tilde{q}_a$ then takes the following form in terms of $\Omega(t)$:

$$\tilde{q}_a = \frac{1}{\alpha}\left(\beta_a - (n \cdot \beta) n_a\right) \sin \Omega(t) + r n_a \cos \Omega(t)\,,$$

where β_a is the N-dimensional vector $\beta_a = (\beta_1, \beta_2, \ldots, \beta_{N-1}, 0)$. Substituting the above result into our original condition $r = \sqrt{\tilde{q} \cdot \tilde{q}}$, leads to

$$r = \sqrt{\frac{\beta^2 - (n \cdot \beta)^2}{\alpha^2}}\,. \tag{8.58}$$

Thus the radius r is a constant of motion and fixed in terms of the $2N$ constant coordinates and momenta. Knowing the gauge invariant observables, we are immediately led to the solutions for the coordinates of the second class system.

Chapter 9

Operator Quantization of Second Class Systems

9.1 Introduction

In chapters 3 and 5 we have discussed in detail the classical Poisson bracket formulation of the Hamilton equations of motion for singular systems, and their symmetries, respectively. We now turn to the problem of formulating the corresponding quantum theory. As we shall show, purely second class systems allow for a straightforward operator quantization (apart from possible ordering problems), while theories which involve also first class constraints must first be effectively converted to second class systems by imposing an appropriate number of gauge conditions. These are subsidiary conditions imposed from the "outside", i.e., they are not part of the Euler-Lagrange equations of motion. When quantizing second class systems we shall be led to introduce an *extended* Hamiltonian which includes all the constraints, primaries and secondaries, with their respective Lagrange multipliers, and - in the case of gauge fixing - the gauge conditions as well. The gauge conditions are chosen in such a way that, together with the second class constraints, they turn the theory into a pure second class system. In terms of the extended Hamiltonian the equations of motion take a form analogous to those expressed in terms of the total Hamiltonian. These will be shown to be completely equivalent to the equations of motion formulated by Dirac.

9.2 Systems with only second class constraints

Consider a purely second class system subject to the constraints $\Omega^{(2)}_{A_2} = 0$. Such a system possesses no local (gauge) symmetry. As we have seen in chapter 3, the classical equations of motion can then be written in form

$$\begin{aligned} \dot{q}_i &= \{q_i, H_0\}_D \ , \\ \dot{p}_i &= \{p_i, H_0\}_D \ , \\ \Omega^{(2)}_{A_2} &= 0 \ , \end{aligned} \tag{9.1}$$

where the second class constraints are implemented strongly by the Dirac brackets. Since the Dirac brackets have the same algebraic properties as the Poisson brackets, any Dirac bracket of two functions can be computed in the standard way from the fundamental Dirac brackets,

$$\{q_i, p_j\}_D = \{q_i, p_j\} - \sum_{A_2,B_2=1}^{N_2} \{q_i, \Omega^{(2)}_{A_2}\} Q^{-1}_{A_2B_2} \{\Omega^{(2)}_{B_2}, p_j\} \ , \tag{9.2}$$

with

$$Q_{A_2B_2} = \{\Omega^{(2)}_{A_2}, \Omega^{(2)}_{B_2}\} \ . \tag{9.3}$$

In the case of a non-singular system, the last term on the rhs of (9.2) is absent, and the transition to the quantum theory is effected by replacing $i\hbar\{q_i, p_j\}$ by the commutator of the corresponding operators (denoted by a "hat"), $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$. For a singular system with second class constraints, this prescription is inconsistent with the constraints. Thus consider for example the Poisson bracket of a second class constraint $\Omega^{(2)}_A$ with any function of the canonical variables. This bracket will in general not vanish on the constrained surface, in contrast to the commutator of the corresponding operators, which by construction does. For the same reason one cannot impose the constraints on the states $|\psi>$. On the other hand, since second class constraints are implemented strongly by the Dirac brackets, this suggests that in the presence of such constraints, the fundamental commutators of the canonical variables are obtained from the corresponding Dirac brackets by the prescription

$$[\hat{q}_i, \hat{p}_j] = i\hbar \widehat{\{q_i, p_j\}}_{\mathcal{D}} \ , \tag{9.4}$$

where the hat over the rhs term means, that first the Dirac bracket is computed, and only then the expression so obtained is replaced by the corresponding operator. Actually this prescription is only well defined if the Dirac bracket on the rhs is a non singular expression of the canonical variables, and modulo possible ordering problems. In any case, the quantum version of the rhs of

(9.4) must be defined in such a way, that the second class constraints are implemented strongly. In many cases of physical interest, the transition will be obvious. Modulo such problems, the quantum equations of motion for a purely second class system then take the form of strong equalities,

$$\begin{aligned}
i\hbar\dot{\hat{q}}_i &= [\hat{q}_i, \hat{H}] \ , \\
i\hbar\dot{\hat{p}}_i &= [\hat{p}_i, \hat{H}] \ , \\
\hat{\Omega}^{(2)}_{A_2} &= 0 \ ,
\end{aligned} \tag{9.5}$$

where the commutators are obtained in the usual way from the fundamental commutators (9.4).

9.3 Systems with first and second class constraints

Let us next turn to the case where the Lagrangian also leads to first class constraints $\Omega^{(1)}_{A_1}$, ($A_1 = 1, \cdots, N_1$), i.e., the system exhibits a local symmetry, or "gauge invariance". The quantization of such a system is more subtle. The number of physical degrees of freedom is now further reduced by the number of first class constraints. In this case it is convenient to implement the second class constraints strongly via Dirac brackets and to include *all* first class constraints explicitly in the equations of motion. This does not affect the dynamics of gauge invariant observables. The classical equations of motion then read

$$\begin{aligned}
\dot{q}_i &= \{q_i, H\}_{\mathcal{D}} + \xi^{A_1}\{q_i, \Omega^{(1)}_{A_1}\} \ , \\
\dot{p}_i &= \{p_i, H\}_{\mathcal{D}} + \xi^{A_1}\{p_i, \Omega^{(1)}_{A_1}\} \ , \\
\Omega^{(1)}_{A_1} &= 0 \ ; \quad \Omega^{(2)}_{A_2} = 0 \ ,
\end{aligned} \tag{9.6}$$

where the Dirac brackets are constructed from the second class constraints. Correspondingly, the equation of motion for any function of the canonical variables reads

$$\dot{f}(q,p) \approx \{f, H\}_D + \xi^{A_1}\{f, \Omega^{(1)}_{A_1}\} \ . \tag{9.7}$$

Note that the parameters $\{\xi^{A_1}\}$, reflecting the gauge degrees of freedom, are not fixed by the requirement of persistence in time of the constraints, but are left undetermined. Since the Poisson bracket of any function $f(q,p)$ with a first class constraint is weakly equivalent to the corresponding Dirac bracket, eq. (9.7) can be replaced by

$$\dot{f} \approx \{f, H\}_{\mathcal{D}} + \xi^{A_1}\{f, \Omega^{(1)}_{A_1}\}_{\mathcal{D}} \ , \tag{9.8}$$

so that the second class constraints are now implemented strongly, while the first class constraints continue to be implemented weakly. One would now naively expect that on quantum level the Dirac backets, multiplied by $i\hbar$, are all replaced by the corresponding commutators. While this prescription is consistent with the strong implementation of the second class constraints on operator level, this is not the case for first class constraints. In fact, since $\{f, \Omega^{(1)}_{A_1}\}_{\mathcal{D}}$ does not vanish in general on the subspace $\Omega^{(1)}_{A_1} = 0$, the transition to the commutator is not only inconsistent with implementation of the constraints on operator level, but also inconsistent with their implementation on the states $|\Psi>$, i.e. $\hat{\Omega}^{(1)}_{A_1}|\Psi>= 0$, as seen by considering matrix elements $<\Psi'|[f, \Omega^{(1)}_{A_1}]|\Psi>$. However, as we show below, one can nevertheless formulate an operator valued quantum theory, by introducing a suitable set of subsidary (gauge fixing) conditions, effectively turning the mixed constrained system into a purely second class one. There is however a difference between a purely second class system and a gauge fixed theory. While in the former case all the constraints are part of the dynamical equations generated by a given Lagrangian, the gauge fixing conditions are intoduced from the outside. Gauge variant quantities will of course depend on the choice of gauge, while gauge invariant quantities (observables) do not. This is also evident in (9.8), since in this case the Dirac bracket of $f(q,p)$ with $\Omega^{(1)}_{A_1}$ vanishes weakly, implying independence of the parameters ξ^{A_1}. We now present some details.

Consider the equations of motion (9.6) written entirely in terms of Poisson brackets,

$$
\begin{aligned}
\dot{q}_i &\approx \{q_i, H^{(1)}\} + \xi^{A_1}\{q_i, \Omega^{(1)}_{A_1}\} \,, \\
\dot{p}_i &\approx \{p_i, H^{(1)}\} + \xi^{A_1}\{p_i, \Omega^{(1)}_{A_1}\} \,,
\end{aligned}
\tag{9.9}
$$

where $H^{(1)}$ has been defined in (3.65), [1]

$$
H^{(1)} = H - \sum_{A_2}^{N_2} \Omega^{(2)}_{A_2} Q^{-1}_{A_2 B_2}\{\Omega^{(2)}_{B_2}, H\} \,.
$$

Since the dynamics of observables on Γ does not depend on the n_1 parameters ξ^{A_1}, we fix these parameters by introducing n_1 suitable (gauge) conditions, where n_1 is the number of first class constraints. Let $\chi_{A_1}(q,p) = 0$ $(A_1 = 1, \cdots, n_1)$ be such a set of gauge conditions. If these conditions are to fix the gauge completely, then the vanishing of the variation induced on χ_{A_1} by the first class constraints, i.e.

$$
\delta\chi_{A_1} = \epsilon^{B_1}(t)\{\chi_{A_1}, \Omega^{(1)}_{B_1}\} = 0 \,,
$$

[1] Here H is understood to be weakly equivalent to the canonical Hamiltonian evaluated on the primary surface, H_0.

must necessarily imply the vanishing of the parameters $\{\epsilon^{B_1}\}$. Hence the determinant of the square matrix with elements

$$\Delta_{A_1B_1} = \{\chi_{A_1}, \Omega^{(1)}_{B_1}\} \tag{9.10}$$

must be different from zero, i.e.,

$$\det\ \Delta \neq 0\,. \tag{9.11}$$

The gauge conditions must therefore be chosen such that (9.11) holds. They must also be consistent with the equations of motion (9.9), augmented by the gauge conditions:

$$\dot{f} = \{f, H^{(1)}\} + \xi^{A_1}\{f, \Omega^{(1)}_{A_1}\}\ , \tag{9.12}$$

$$\Phi_r := (\Omega^{(1)}, \Omega^{(2)}, \chi) = 0\,, \quad r = 1, \cdots, 2n_1 + N_2\ . \tag{9.13}$$

Persistence in time of the subsidiary conditions $\chi_{A_1} = 0$ now requires that [2]

$$\dot{\chi}_{A_1} \approx \{\chi_{A_1}, H^{(1)}\} + \xi^{B_1}\{\chi_{A_1}, \Omega^{(1)}_{B_1}\} \approx 0\ .$$

Because of (9.11), it follows that the gauge parameters are determined:

$$\xi^{A_1} \approx -\sum_{B_1} \Delta^{-1}_{A_1B_1}\{\chi_{B_1}, H^{(1)}\}\ .$$

We therefore have that

$$\dot{f} \approx \{f, H_{gf}\}\quad , \tag{9.14}$$

where

$$H_{gf} = H^{(1)} - \sum_{A_1,B_1} \Omega^{(1)}_{A_1}\Delta^{-1}_{A_1B_1}\{\chi_{B_1}, H^{(1)}\} \tag{9.15}$$

is the gauge fixed Hamiltonian. Hence in the *gauge invariant sector* the equations (9.14) are equivalent to (9.7), with the parameters ξ^{A_1} fixed by the gauge conditions $\chi_{A_1} = 0$. These gauge conditions select from all possible trajectories one representative from each gauge orbit (assuming there exists no Gribov ambiguity [Gribov 1978]). We now prove the following theorem :

Theorem:

In the gauge invariant sector the equations of motion (9.8) are completely equivalent to the set

$$\dot{f} \approx \{f, H\}_{\mathcal{D}^*}\quad ;\quad \Phi_r = 0\ ,\ \forall r \tag{9.16}$$

[2] The weak equality now includes all the constraints and gauge conditions, $\Phi_r = 0$.

where
$$\{A, B\}_{\mathcal{D}^*} = \{A, B\} - \sum_{r,s} \{A, \Phi_r\} Q^{*-1}_{rs} \{\Phi_s, B\} \;, \tag{9.17}$$
$$Q^*_{rs} = \{\Phi_r, \Phi_s\} \;, \tag{9.18}$$
and
$$\Phi_r := (\chi, \Omega^{(1)}, \Omega^{(2)}) \;.$$
Note that now the Dirac bracket involves all the constraints as well as gauge conditions, which together form a second class system.

Proof

We first show that the equations of motion (9.14) are fully equivalent to
$$\dot{f} = \{f, \bar{H}_E\} \,, \quad \Phi_r = 0 \,, \quad \forall r \tag{9.19}$$
where
$$\bar{H}_E = H + \eta^{A_2}\Omega^{(2)}_{A_2} + \xi^{A_1}\Omega^{(1)}_{A_1} + \zeta^{A_1}\chi_{A_1} \tag{9.20}$$
is now the fully extended Hamiltonian, which includes *all* constraints and gauge conditions (multiplied by Lagrange multipliers). To prove the above claim we examine the implications of the consistency conditions $\dot{\Phi}_r = 0$, where the time evolution is now generated by $\bar{H}_E$:
$$a)\ \dot{\Omega}^{(1)}_{A_1} \approx \{\Omega^{(1)}_{A_1}, \chi_{B_1}\}\zeta^{B_1} \approx 0 \;,$$
$$b)\ \dot{\Omega}^{(2)}_{A_2} \approx \{\Omega^{(2)}_{A_2}, H\} + \{\Omega^{(2)}_{A_2}, \Omega^{(2)}_{B_2}\}\eta^{B_2} + \{\Omega^{(2)}_{A_2}, \chi_{B_1}\}\zeta^{B_1} \approx 0 \;,$$
$$c)\ \dot{\chi}_{A_1} \approx \{\chi_{A_1}, H\} + \{\chi_{A_1}, \Omega^{(2)}_{B_2}\}\eta^{B_2} + \{\chi_{A_1}, \Omega^{(1)}_{B_1}\}\xi^{B_1} + \{\chi_{A_1}, \chi_{B_1}\}\zeta^{B_1} \approx 0 \;.$$
In a) we used the fact that the Poisson bracket of $\Omega^{(1)}_{A_1}$ with each of the first three terms on the rhs of (9.20) vanishes on the constrained surface. [3] From a) and (9.11) it follows, that
$$\zeta^{B_1} \approx 0 \;. \tag{9.21}$$
Hence the term $\zeta^{A_1}\chi_{A_1}$ in H_E does not contribute to the equations of motion (9.19). It then follows from b) that
$$\eta^{A_2} \approx -\sum_{B_2} Q^{-1}_{A_2B_2}\{\Omega^{(2)}_{B_2}, H\} \;, \tag{9.22}$$

[3] Recall that the constraints $\Omega^{(1)}_{A_1}$ and $\Omega^{(2)}_{A_2}$ were generated by the Dirac algorithm which involves the total Hamiltonian constructed from the canonical Hamiltonian and the primary constraints only. Hence the $\Omega^{(1)}_{A_1}$'s satisfy, in particular, the following equations of motion: $\dot{\Omega}^{(1)}_{A_1} \approx \{\Omega^{(1)}_{A_1}, H_T\} \approx \{\Omega^{(1)}_{A_1}, H\} \approx 0$.

where $Q_{A_2B_2}$ has been defined in (9.3). With (9.21) and (9.22), c) reduces to

$$\{\chi_{A_1}, H^{(1)}\} + \{\chi_{A_1}, \Omega^{(1)}_{B_1}\}\xi^{B_1} \approx 0 \ ,$$

with the solution

$$\xi^{A_1} \approx -\sum_{B_1} \Delta^{-1}_{A_1B_1}\{\chi_{B_1}, H^{(1)}\} \ , \tag{9.23}$$

where $\Delta_{A_1B_1}$ has been defined in (9.10). Inserting the results (9.21), (9.22), and (9.23) in (9.20), we see that $\bar{H}_E$ can be effectively replaced by (9.15), thus proving the equivalence of (9.14) with (9.19), and therefore also with (9.7) within the gauge invariant sector.

It is now an easy matter to prove the equivalence with (9.16). To this effect we write (9.20) in the compact form

$$\bar{H}_E = H + \rho^r \Phi_r \ , \tag{9.24}$$

where $\rho = (\eta, \chi, \zeta)$. The consistency equations for the constraints and gauge conditions read

$$\dot{\Phi}_r \approx \{\Phi_r, \bar{H}_E\} \approx 0 \ .$$

Since the constraints $\{\Phi_r = 0\}$ form a second class system with $det\{\Phi_r, \Phi_s\} = 0$, we have

$$\rho^r \approx -\sum_{r'} Q^{*-1}_{rr'}\{\Phi_{r'}, H\} \ .$$

Inserting this expression in (9.24) one is immediately led to (9.16), which is now a suitable starting point for making the transition to the quantum theory. Indeed, since all constraints and gauge conditions are now implemented strongly by the $\mathcal{D}^*$-bracket, the transition to the quantum theory is effected by replacing the $\mathcal{D}^*$-bracket, multiplied by $i\hbar$, by the corresponding commutator,

$$[\hat{A}, \hat{B}] = i\hbar\widehat{\{A, B\}}_{\mathcal{D}^*} \ .$$

Clearly the form of the equation of motion will depend on the chosen gauge. On the other hand, the equation of motion for observables, i.e. those functions whose Poisson bracket with the first class constraints vanish weakly, are independent of the choice of gauge.

9.3.1 Example: the free Maxwell field in the Coulomb gauge

We now illustrate the above procedure by an example. Consider once more the free Maxwell field. As we have seen in example 4 of chapter 3, this theory only possesses two first class constraints (one primary and one secondary). In order to fix the gauge completely we therefore need two subsidiary conditions satisfying the requirement (9.11). Since we want to realize the Coulomb gauge, one of these conditions is clearly $\vec{\nabla}\cdot\vec{A}=0$. For the other condition we choose $A^0=0$. [4] This leads to a non-vanishing determinant for the matrix (9.10). Let us label the constraints and gauge conditions as follows:

$$\Phi_1 \equiv \pi_0 = 0\ , \quad \Phi_2 \equiv \partial^j \pi_j = 0\ ,$$
$$\Phi_3 \equiv A^0 = 0\ , \quad \Phi_4 \equiv \partial_j A^j = 0\ .$$

These conditions form a second class system of constraints. The non-vanishing matrix elements $Q^*_{rr'}$ defined in (9.18) are then given by

$$Q^*_{13}(\vec{x},\vec{y}) = -Q^*_{31}(\vec{y},\vec{x}) = -\delta(\vec{x}-\vec{y})\ ,$$
$$Q^*_{24}(\vec{x},\vec{y}) = -Q^*_{42}(\vec{y},\vec{x}) = -\nabla^2_x\delta(\vec{x}-\vec{y})\ .$$

The inverse of the matrix $Q^*_{rr'}(\vec{x},\vec{y})$ is defined by

$$\sum_{r"}\int d^3z\ Q^{*-1}_{rr"}(\vec{x},\vec{z})Q^*_{r"r'}(\vec{z},\vec{y}) = \delta_{rr'}\delta(\vec{x}-\vec{y})\ .$$

The non-vanishing matrix elements of Q^{*-1} are easily calculated to be

$$Q^{*-1}_{13}(\vec{x},\vec{y}) = -Q^{*-1}_{31}(\vec{y},\vec{x}) = \delta(\vec{x}-\vec{y})\ ,$$
$$Q^{*-1}_{24}(\vec{x},\vec{y}) = -Q^{*-1}_{42}(\vec{y},\vec{x}) = \frac{1}{\nabla^2}\delta(\vec{x}-\vec{y})\ , \tag{9.25}$$

where $\frac{1}{\nabla^2}$ is defined by

$$\frac{1}{\nabla^2}f(\vec{x}) \equiv \int d^3z\ G(\vec{x},\vec{z})f(\vec{z})\ ,$$

with $G(\vec{x},\vec{z})$ the Green function of the Laplace operator,

$$\nabla^2 G(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y})\,.$$

Consider now, e.g., the $\mathcal{D}^*$-bracket (9.17), generalized to a system with an infinite number of degrees of freedom. We have

$$\{A^i(\vec{x},t),\pi_j(\vec{y},t)\}_{\mathcal{D}^*} = \{A^i(\vec{x},t),\pi_j(\vec{y},t)\}$$
$$-\sum_{rr'}\int d^3z d^3z'\{A^i(\vec{x},t),\Phi_r(\vec{z},t)\}Q^{*-1}_{rr'}(\vec{z},\vec{z}')\{\Phi_{r'}(\vec{z}'.t),\pi_j(\vec{y},t)\}\ .$$

[4] On the Lagrangian level these subsidiary conditions would be inconsistent in the presence of sources. Within a Hamiltonian formulation, however, this is an allowed choice, since on the level of the extended Hamiltonian any choice of A^0 can be absorbed by the Lagrange multiplier associated with the secondary (Gauss law) constraint, to be determined by the subsidiary conditions.

Making use of (9.25) one verifies that

$$\{A^i(\vec{x},t),\pi_j(\vec{y},t)\}_{\mathcal{D}^*} = (\delta^i_j - \frac{\partial^i\partial_j}{\nabla^2})\delta(\vec{x}-\vec{y})\,. \tag{9.26}$$

Note that the Coulomb gauge and the secondary constraint $\partial^i\pi_i = 0$ are implemented strongly, as expected. In the same way one finds that

$$\{A^i(\vec{x},t),A^j(\vec{y},t)\}_{\mathcal{D}^*} = 0\,, \quad \{\pi_i(\vec{x},t),\pi_j(\vec{y},t)\}_{\mathcal{D}^*} = 0\,.$$

Furthermore, any $\mathcal{D}^*$-bracket involving A^0 or π_0 vanishes identically. On quantum level, (9.26) translates to

$$[\hat{A}^i(\vec{x},t),\hat{\pi}_j(\vec{y},t)] = i\hbar(\delta^i_j - \frac{\partial^i\partial_j}{\nabla^2})\delta(\vec{x}-\vec{y})\,.$$

This commutator is of course well known, but was constructed here within the Hamiltonian framework. [5]

Summarizing we have: the commutators of the phase-space variables are determined from the $\mathcal{D}^*$-brackets constructed here from the first class constraints and corresponding gauge conditions. These are thereby implemented strongly. Hence the corresponding *operators* can be set equal to the null operator. In general the realization of the operators satisfying the correct commutation relations, as dictated by the $\mathcal{D}^*$-brackets, may however be difficult, if not impossible.

9.3.2 Concluding remark

So far we have considered second class systems, including the case of gauge fixed mixed systems. The basis for their quantization was always provided by their Dirac bracket formulation, where the constraints and gauge conditions were implemented strongly. Apart from ordering problems and possible singularities, this allowed for an operator realization of the equations of motion.

For observables, i.e. gauge invariant quantities, Dirac has proposed an alternative way of dealing with first class or mixed systems on operator level [Dirac 1964]. The first class constraints are imposed as conditions on the physical states in the form [6]

$$\hat{\Omega}^{(1)}_{A_1}|\Psi>= 0\,. \tag{9.27}$$

[5] For less trivial examples see e.g., [Girotti 1982, Kiefer 1985].

[6] If second class constraints are present then we assume that they have been strongly implemented by replacing the Poisson brackets by Dirac brackets constructed from the second class constraints.

The states are therefore also gauge invariant. Hence observables as well as states depend effectively only on gauge invariant combinations of the phase space variables. This ensures on operator level, that for two gauge invariant states $|\Psi>$ and $|\Phi>$

$$< \Psi|[\hat{\mathcal{O}}, \hat{\Omega}^{(1)}_{A_1}]|\Phi >= 0\,, \quad A_1 = 1, \cdots, N_1 \ .$$

In the case of purely first class systems we then have a canonical Poisson bracket structure. In Quantum Mechanics this means that the operators $\hat{\Omega}^{(1)}_{A_1}$ can be realized by making the substitution $p_i \to \frac{\hbar}{i}\frac{\partial}{\partial q_i}$ in the corresponding classical expression, so that the condition (9.27) takes the form

$$\hat{\Omega}^{(1)}_{A_1}\left(q, \frac{\hbar}{i}\vec{\nabla}\right)\psi(q,t) = 0 \ ,$$

where $\psi(q,t)$ is the wave function in configuration space corresponding to the state $|\Psi>$. This is a differential equation, whose solution is the family of all gauge invariant wave functions.

The actual realization of Dirac's method may however not be straightforward. Thus the definition of the operators $\hat{\Omega}^{(1)}_{A_1}$ may be problematic due to ordering problems. Furthermore one in general also seeks to preserve the classical algebraic properties of the constraints, which may also be problematic.

Another method consists in quantizing only gauge invariant degrees of freedom constructed from the q_i's and p_i's, and all observables are expressed in terms of these. Hence no gauge conditions are imposed. This reduced phase space is endowed with the standard Poisson bracket structure. For a systematic method of arriving at such a reduced phase space formulation of the dynamics we refer the reader to the last section of chapter 4. Correlation functions of observables (which only depend on the star-variables) will have the standard path integral representation for an unconstrained system. But the price paid for such a formulation may be too high to be useful. For example, in a gauge theory like the Maxwell theory, the star-variables are given by the *non-local* (gauge invariant) expressions for the transverse potentials and their corresponding conjugate momenta. It is therefore of interest to develop non-operator methods based on functional techniques. This will be the subject of the following three chapters.

Chapter 10

Functional Quantization of Second Class Systems

10.1 Introduction

The following discussion applies to purely second class systems, or to mixed systems, where the gauge degrees of freedom are frozen by imposing a set of gauge conditions. In this case the latter system is also effectively second class. As we have seen in the previous chapter the dynamics is then conveniently described by the extended Hamiltonian, which includes all the constraints and gauge conditions. This allowed an operator quantization once Dirac brackets were introduced.

For second class systems there exist standard methods for obtaining the functional representation of the partition function. In this chapter we show how this is achieved. As we will see, the often non-polynomial structure of the Dirac brackets reflects itself in the functional integration measure, and makes a perturbative treatment in such gauges difficult, if not impossible. In this case one therefore seeks a formulation based on canonical Poisson brackets. Such a formulation can be obtained by introducing additional ghost degrees of freedom and will be the subject of the following chapter. In the present chapter the partition function will be expressed as an integral over the original Lagrangian coordinates and their conjugate momenta [Leibbrandt 1987].

10.2 Partition function for second class systems

Consider the general case of a mixed system in a fixed gauge, with the action given by

$$S_{gf} = \int dt(\sum_i p_i \dot{q}_i - H - \xi^r \Phi_r) \ , \tag{10.1}$$

where the sum over "i" includes all coordinates and momenta, and where we have collected all the constraints Ω_A and gauge fixing functions χ_{A_1} into a single "vector" as in (9.13), with $N_2 = 2n_2$, since the number of second class constraints is always even. The equations of motion derived from the action principle $\delta S_{gf} = 0$ (where the variation includes the variation with respect to ξ_r) read

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} + \xi^r \frac{\partial \Phi_r}{\partial p_i} \ , \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} - \xi^r \frac{\partial \Phi_r}{\partial q_i} \ , \\ \Phi_r &= 0 \ \ ; \ \ r = 1, \dots, 2n \ , \end{aligned} \tag{10.2}$$

where $n = n_1 + n_2$. Note the close analogy with the familiar Lagrangian method of the second kind in classical mechanics for implementing a set of given *external* constraints. The solutions to (10.2) also correspond to an extremum of the action $S = \int dt(\sum_i p_i \dot{q}_i - H)$ with the q's and p's restricted to the full constrained surface Γ^*.

We next show that for gauge fixing functions χ_A satisfying $\{\chi_{A_1}, \chi_{B_1}\} = 0$, the partition function is given by

$$Z = \int DqDp \prod_{r,t} \delta(\Phi_r(q,p)) \prod_t \sqrt{|det\{\Phi_r, \Phi_{r'}\}|} e^{i \int dt [\sum_i p_i \dot{q}_i - H(q,p)]} \ . \tag{10.3}$$

To this effect let us introduce a new set of variables of which a subset consists of the constraints Φ_r. Consider the Poisson bracket of these constraints,

$$\{\Phi_r, \Phi_{r'}\} = Q_{rr'}(q,p) \ ; \ r = 1, \dots, 2n \ . \tag{10.4}$$

Here Q is an antisymmetric matrix. Hence there exists an orthogonal matrix V such that

$$VQV^T = \tilde{Q} \,, \tag{10.5}$$

where $\tilde{Q}$ takes the following off diagonal form [Senjanovic 1976]

$$\tilde{\mathbf{Q}} = \begin{pmatrix} & \epsilon \\ -\epsilon & \end{pmatrix} \ ,$$

where

$$\epsilon = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \cdot & & \\ \cdot & & & \\ 1 & & & \end{pmatrix} .$$

Defining the following linear combinations of constraints

$$\tilde{\Phi}_r(q,p) = \sum_{r'} V_{rr'}(q,p)\Phi_{r'}(q,p) \tag{10.6}$$

we have weakly

$$\{\tilde{\Phi}_r, \tilde{\Phi}_{r'}\} \approx \tilde{Q}_{rr'} \ . \tag{10.7}$$

Hence the variables

$$\begin{aligned} \tilde{q}_\ell &= \tilde{\Phi}_\ell \quad ; \ \ell = 1, \cdots, n \ , \\ \tilde{p}_\ell &= \tilde{\Phi}_{2n-\ell+1} \quad ; \ \ell = 1, \cdots, n \end{aligned} \tag{10.8}$$

form canonical pairs in the *weak* sense, i.e.

$$\{\tilde{q}_\ell, \tilde{p}_{\ell'}\} \approx \delta_{\ell\ell'} \tag{10.9}$$

where "weak" includes the constraints and gauge conditions. The weak equality results from the fact that the matrix $V(q,p)$ is in general a function of the canonical variables, in which case it does not commute with the Poisson bracket operation. The remaining variables $q^* \equiv \{q^*_a\}$, $p^* \equiv \{p^*_a\}$, labeling the points on Γ^*, are then chosen, if possible, so that the q^*'s, p^*'s, $\tilde{q}$'s and $\tilde{p}$'s form a canonical set with respect to q and p in the weak sense.

Note that if the matrix $Q_{rr'}$, and hence $V_{rr'}$, is independent of q, p, then the weak equality in (10.9) is turned into a strong equality. In the case where $Q_{rr'}(q,p)$ depends on the phase space variables, one can nevertheless proceed iteratively in order to implement the canonical commutation relations strongly [Senjanovich 1976]. In some special cases, a closed solution can be constructed. In general, however, this iterative construction is extremely cumbersome. But since the integrations in the partition function will be clearly restricted to the constrained surface Γ^*, a strong implementation of the commutation relations (10.9) is not required.

The transformation from the original variables to the new set of variables leaves the measure invariant up to a Jacobian which reduces to the identity on Γ^*. Since this transformation is weakly canonical, we have that

$$\sum_i p_i \dot{q}_i \approx \sum_a p^*_a \dot{q}^*_a + \sum_\ell \tilde{p}_\ell \dot{\tilde{q}}_\ell + \frac{d}{dt} F \, .$$

Next we rewrite the δ-functions in (10.3) in terms of the new set of variables. From (10.6) it follows that

$$\prod_r \delta(\tilde{\Phi}_r) = \frac{1}{|detV|}\prod_r \delta(\Phi_r) \ .$$

Furthermore, from (10.5) and (10.4) we have that

$$|det\ V| = \frac{1}{\sqrt{|det\{\Phi_r, \Phi_{r'}\}|}} \ ,$$

where we made use of the fact that $|det\ \tilde{Q}| = 1$. Hence the ansatz (10.3) leads to the following alternative expression:

$$Z = \int Dq^* Dp^* D\tilde{q} D\tilde{p} \prod_\ell \delta(\tilde{q}_\ell)\delta(\tilde{p}_\ell) e^{i\int dt\ \left(\sum_a p_a^* \dot{q}_a^* - H^*(q^*,p^*)\right)} \ ,$$

where

$$H^*(q^*, p^*) = H(q(q^*, p^*, \tilde{\Phi}), p(q^*, p^*, \tilde{\Phi}))|_{\tilde{\Phi}=0} \, .$$

Upon integrating this expression over $\tilde{q}$ and $\tilde{p}$ we are led to the partition function

$$Z = \int Dq^* Dp^* e^{i\int dt\ \left(\sum_a p_a^* \dot{q}_a^* - H^*(q^*,p^*)\right)} \ .$$

This is the expected result for the partition function of an unconstrained system. Thus the variation of the action leads, as expected, to

$$\dot{q}_a^* = \frac{\partial H^*}{\partial p_a^*} = \{q_a^*, H^*\}_{q^*p^*} \ , \tag{10.10}$$

$$\dot{p}_a^* = -\frac{\partial H^*}{\partial q_a^*} = \{p_a^*, H^*\}_{q^*p^*} \ . \tag{10.11}$$

We now show that these equations are indeed equivalent to (10.2). To this effect we compute $\dot{q}_i$ and $\dot{p}_i$, with the q_i's and p_i's expressed in terms of the new variables.

$$\dot{q}_i = \frac{\partial q_i}{\partial \tilde{q}_\ell}\dot{\tilde{q}}_\ell + \frac{\partial q_i}{\partial \tilde{p}_\ell}\dot{\tilde{p}}_\ell + \frac{\partial q_i}{\partial q_a^*}\dot{q}_a^* + \frac{\partial q_i}{\partial p_a^*}\dot{p}_a^* \ ,$$

with a similar expression for $\dot{p}_i$. On the constrained surface the first two terms vanish, since the constraints hold for all times. Hence, making use of (10.10) and (10.11), we conclude that

$$\dot{q}_i \approx \{q_i, H^*\}_{q^*p^*} \approx \{q_i, H\}_{q^*p^*} \ .$$

The rhs, we claim, is nothing but the corresponding $\mathcal{D}^*$-Dirac bracket $\{q_i, H\}_{\mathcal{D}^*}$ evaluated on Γ^*; i.e., for $\tilde{p}_\ell = \tilde{q}_\ell = 0$:

$$\dot{q}_i \approx \{q_i, H\}_{\mathcal{D}^*} \ .$$

This follows from the weak equality, [1]

$$\{A, B\}_{\mathcal{D}^*} \approx \{A, B\}_{q^* p^*} \ , \tag{10.12}$$

where the $\mathcal{D}^*$-bracket is given by

$$\{A, B\}_{\mathcal{D}^*} = \{A, B\} - \sum_{r,r'} \{A, \tilde{\Phi}_r\} \tilde{Q}^{-1}_{rr'} \{\tilde{\Phi}_{r'}, B\} \ , \tag{10.13}$$

and the Poisson bracket in (10.12) is calculated with respect to the star variables.

The proof of (10.12) makes use of the following property of Poisson brackets under a canonical transformation $q, p \to Q, P$:

$$\{A, B\}_{q,p} = \{A, B\}_{QP} \ . \tag{10.14}$$

Consider the Dirac bracket (10.13). The inverse of the matrix (10.7) is given by $-\tilde{Q}$. Hence with the identifications (10.8), the bracket (10.13) is given by

$$\{A, B\}_{\mathcal{D}^*} = \{A, B\} + \sum_\ell \{A, \tilde{q}_\ell\}\{\tilde{p}_\ell, B\} - \sum_\ell \{A, \tilde{p}_\ell\}\{\tilde{q}_\ell, B\} \,. \tag{10.15}$$

But because of (10.14) we can now replace in a weak sense all Poisson brackets (with respect to q and p) by the corresponding Poisson brackets with respect to $\tilde{q}$, $\tilde{p}$, q^* and p^*. Hence $\{A, \tilde{q}_\ell\} \approx -\frac{\partial A}{\partial \tilde{p}_\ell}$, $\{A, \tilde{p}_\ell\} \approx \frac{\partial A}{\partial \tilde{q}_\ell}$, $\{\tilde{q}_\ell, B\} \approx \frac{\partial B}{\partial \tilde{p}_\ell}$, $\{\tilde{p}_\ell, B\} \approx -\frac{\partial B}{\partial \tilde{q}_a}$, so that the sums on the rhs of (10.15) reduce weakly to $-\{A, B\}_{\tilde{q}\tilde{p}}$. On the other hand

$$\{A, B\} \equiv \{A, B\}_{qp} \approx \{A, B\}_{q^* p^*} + \{A, B\}_{\tilde{q}\tilde{p}} \,.$$

We thus arrive at the weak equality (10.12).

Let us now return to the partition function (10.3) and write it in an alternative form which is valid if the gauge conditions $\chi_{A_1} = 0$ have been chosen so that their Poisson brackets vanish on Γ^*. [2] Consider the matrix (10.4). It has the form,

$$\mathbf{Q} = \begin{pmatrix} \{\chi, \chi\} & \{\chi, \Omega^{(1)}\} & \{\chi, \Omega^{(2)}\} \\ \{\Omega^{(1)}, \chi\} & \{\Omega^{(1)}, \Omega^{(1)}\} & \{\Omega^{(1)}, \Omega^{(2)}\} \\ \{\Omega^{(2)}, \chi\} & \{\Omega^{(2)}, \Omega^{(1)}\} & \{\Omega^{(2)}, \Omega^{(2)}\} \end{pmatrix} \ .$$

[1] We use the same symbol for a function considered to be either a function of q, p, or of the new (weakly) canonical set $\tilde{q}$, $\tilde{p}$, $q^\star$, $p^\star$.

[2] For example, the Coulomb gauge in QED and QCD has this property.

On the constrained surface $\mathbf{Q}$ reduces to:

$$\mathbf{Q}|_{\Gamma^*} = \{\tilde{\Phi}_r, \tilde{\Phi}_{r'}\}|_{\Gamma^*} = \begin{pmatrix} 0 & \{\chi, \Omega^{(1)}\} & \{\chi, \Omega^{(2)}\} \\ \{\Omega^{(1)}, \chi\} & 0 & 0 \\ \{\Omega^{(2)}, \chi\} & 0 & \{\Omega^{(2)}, \Omega^{(2)}\} \end{pmatrix} .$$

For this particular form of the matrix we have

$$|\det\{\Phi_r, \Phi_{r'}\}| \approx \left(\det\{\chi_{A_1}, \Omega^{(1)}_{B_1}\}\right)^2 \cdot |\det\{\Omega^{(2)}_{A_2}, \Omega^{(2)}_{B_2}\}| .$$

The partition function (10.3) can therefore be also written in the form [Fradkin 1977]

$$Z = \int DqDp \prod_{r,t} \delta(\Phi_r(q,p)) \prod_t |\det\{\chi_{A_1}, \Omega^{(1)}_{B_1}\}| \prod_t \sqrt{|\det\{\Omega^{(2)}_{A_2}, \Omega^{(2)}_{B_2}\}|} \times e^{i\int dt\, \left(\sum_i p_i \dot{q}_i - H(q,p)\right)} . \quad (10.16)$$

Hence the naive form for the partition function is modified by the presence of two determinants in the measure constructed from the Poisson brackets of the second class constraints, and from the Poisson brackets of the first class constraints with the gauge conditions.

Example

Consider the Lagrangian (7.17). The constraints are second class and are given in the notation of (10.8) by

$$\tilde{q} = p_y \ , \quad \tilde{p} = \frac{1}{2}(p_x + x - 2y) \ , \qquad (10.17)$$

where the factor 1/2 has been introduced for convenience. With this factor these constraints form a canonical pair in the strong sense, so that the matrix V in (10.6) is just the unit matrix. We next introduce the variables q^* and p^* as follows [3]

$$q^* = x + p_x; \quad p^* = p_x + \frac{1}{2} p_y \ ,$$

which again form a canonical pair in the strong sense and commute with the constraints. Hence in this example, all equations derived in the previous section will actually hold in the strong sense. In terms of the new variables, the original

[3] Notice that our choice here is different from (8.27).

variables are given by

$$\begin{aligned} x &= q^* - p^* + \frac{1}{2}\tilde{q} \ , \quad p_x = p^* - \frac{1}{2}\tilde{q} \ , \\ y &= \frac{1}{2}q^* - \tilde{p} \ , \quad p_y = \tilde{q} \ . \end{aligned}$$

One verifies that the variables $\tilde{q}$, $\tilde{p}$, q^* and p^* form a canonical set, and that

$$(p_x\dot{x} + p_y\dot{y}) = p^*\dot{q}^* + \tilde{p}\dot{\tilde{q}} + \frac{d}{dt}F \ ,$$

where

$$F = \frac{1}{2}p^*\tilde{q} - \frac{1}{2}p^{*2} - \frac{1}{8}\tilde{q}^2 - \tilde{q}\tilde{p} \ .$$

Furthermore we have that

$$\{x, y\}_{q^*p^*} = \frac{1}{2} \ , \quad \{x, p_x\}_{q^*p^*} = 1 \, .$$

Let us compare these Poisson brackets with the corresponding Dirac brackets defined in (10.13). The only non-vanishing matrix elements of $\tilde{Q}^{-1}$ are $\tilde{Q}^{-1}_{12} = -\tilde{Q}^{-1}_{21} = -1$. One then finds that

$$\{x, y\}_{q^*p^*} = \{x, y\}_{\mathcal{D}^*} \ , \quad \{x, p_x\}_{q^*p^*} = \{x, p_x\}_{\mathcal{D}^*}$$

in agreement with (10.12), which here holds in the strong sense.

Gauge invariance of the partition function

In the following we consider the case of a purely first class system in a fixed (non-dynamical) gauge, [4] and show that the partition function Z_χ does not depend on the choice of gauge conditions $\chi_{A_1} = 0$. In particular we want to show that

$$Z_{\chi'} = Z_\chi$$

where $\chi'(q, p) = 0$ selects another point (q', p') on the gauge orbit of (q, p), connected to (q, p) by a gauge transformation

$$\begin{aligned} q_i(t) \to q'_i(t) &= q_i(t) + \{q_i, G\} = q_i + \frac{\partial G}{\partial p_i} \\ p_i(t) \to p'_i(t) &= p_i(t) + \{p_i, G\} = p_i - \frac{\partial G}{\partial q_i}, \end{aligned} \qquad (10.18)$$

[4] Gauge theories such as QED and QCD in the Coulomb gauge fall into this class.

where G is the generator of gauge transformations (5.7), and where χ and χ' are related by

$$\chi'(q,p) = \chi(q',p') \ . \tag{10.19}$$

To this end we show that

$$Z_{\chi'} = \int DqDp \prod_{A_1,t} \delta\left(\chi'_{A_1}(q,p)\right) \prod_{A_1,t} \delta\left(\Omega^{(1)}_{A_1}(q,p)\right) \tag{10.20}$$

$$\times \prod_t \ \det\{\chi'_{A_1}(q,p), \Omega^{(1)}_{B_1}(q,p)\} \exp\left(i\int_{t_i}^{t_f} dt\ \left(\sum_i p_i\dot{q}_i - H(q,p)]\right)\right)$$

$$= \int Dq'Dp' \prod_{A_1,t} \delta\left(\chi_{A_1}(q',p')\right) \prod_{A_1,t} \delta\left(\Omega^{(1)}_{A_1}(q',p')\right) \tag{10.21}$$

$$\times \prod_t \ \det\{\chi_{A_1}(q',p'), \Omega^{(1)}_{B_1}(q',p')\} \exp\left(i\int_{t_i}^{t_f} dt\ \left(\sum_i p'_i\dot{q}'_i - H(q',p')\right)\right)$$

It then follows that correlation functions of gauge invariant observables satisfying $f(q',p') = f(q,p)$ take the same value in any two gauges which are related by a gauge transformation.

To prove the above statement we first show that the functional integration measure is invariant under an infinitesimal transformation (10.18). This can be seen as follows. First of all

$$\prod_{i,t} dq'_i(t)dp'_i(t) = J \prod_{i,t} dq_i(t)dp_i(t) \ ,$$

where the Jacobian J is given by

$$J = \prod_t \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \ ,$$

and A, B, C, D are matrices with matrix elements $A_{ij} = \frac{\partial q'_i(t)}{\partial q_j(t)}$, $B_{ij} = \frac{\partial q'_i(t)}{\partial p_j(t)}$, $C_{ij} = \frac{\partial p'_i(t)}{\partial q_j(t)}$, $D_{ij} = \frac{\partial p'_i(t)}{\partial p_j(t)}$. Since from (10.18)

$$\frac{\partial q'_i(t)}{\partial q_j(t)} = \delta_{ij} + \frac{\partial^2 G}{\partial q_j(t)\partial p_i(t)} \ ,$$

etc., it follows that

$$J \approx \prod_t \det\begin{pmatrix} \delta_{ij} + \frac{\partial^2 G}{\partial p_i(t)\partial q_j(t)} & \frac{\partial^2 G}{\partial p_i(t)\partial p_j(t)} \\ -\frac{\partial^2 G}{\partial q_i(t)\partial q_j(t)} & \delta_{ij} - \frac{\partial^2 G}{\partial p_j(t)\partial q_i(t)} \end{pmatrix} = \prod_t \det(\mathbb{1} + M(t)) \ ,$$

or [5]

$$J = \prod_t e^{\ln\, \det(1\!\!1+M)} = e^{\sum_t \mathrm{Tr} \ln\,(1\!\!1+M)} \;.$$

For this infinitesimal transformation $Tr \ln\,(1\!\!1 + M) \simeq TrM = 0$. Hence

$$\mathcal{D}q\mathcal{D}p = \mathcal{D}q'\mathcal{D}p' \;. \tag{10.22}$$

Consider next the change induced in the first class constraints by the above gauge transformation. Making use of (10.18) and of the algebra (5.12) for first class constraints, we have that

$$\begin{aligned} \Omega^{(1)}_{A_1}(q',p') &= \Omega^{(1)}_{A_1}(q,p) + \left(\frac{\partial \Omega^{(1)}_{A_1}}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial \Omega^{(1)}_{A_1}}{\partial p_i} \frac{\partial G}{\partial q_i} \right) \\ &= \Omega^{(1)}_{A_1}(q,p) + \epsilon^{B_1}(t) \{ \Omega^{(1)}_{A_1}, \Omega^{(1)}_{B_1} \} \\ &= \Omega^{(1)}_{A_1}(q,p) + \epsilon^{B_1}(t) U^{C_1}_{A_1 B_1} \Omega^{(1)}_{C_1}(q,p) \\ &= \left(\delta^{C_1}_{A_1} + L^{C_1}_{A_1} \right) \Omega^{(1)}_{C_1}(q,p) \;, \end{aligned} \tag{10.23}$$

where

$$L^{C_1}_{A_1} = U^{C_1}_{A_1 B_1} \epsilon^{B_1}(t) \,.$$

From here it follows that

$$\prod_{A_1,t} \delta \left(\Omega^{(1)}_{A_1}(q',p') \right) = \prod_t [\det(1+L)]^{-1} \prod_{A_1,t} \delta \left(\Omega^{(1)}_{A_1}(q,p) \right) \;.$$

Furthermore from (10.23) and (10.19) it follows that we have the weak equality

$$\prod_t \det\{ \Omega^{(1)}_{A_1}(q',p'), \chi_{B_1}(q',p') \} \approx \prod_t \det\{ \Omega^{(1)}_{A_1}(q,p), {\chi'}_{B_1}(q,p) \} \det(1+L)$$

so that, because of the δ-functions, we arrive at the strong equality

$$\begin{aligned} \prod_t \det\{ \Omega^{(1)}_{A_1}(q,p), {\chi'}_{B_1}(q,p) \} \prod_{C_1} \delta \left(\Omega^{(1)}_{C_1}(q,p) \right) = \prod_t \det\{ \Omega^{(1)}_{A_1}(q',p'), \chi_{B_1}(q',p') \} \\ \times \prod_{C_1} \delta \left(\Omega^{(1)}_{C_1}(q',p') \right) \;. \end{aligned} \tag{10.24}$$

Finally we show that, as expected, the action is form invariant under the infinitesimal transformation (10.18). Thus

$$\begin{aligned} H(q',p') &= H(q,p) + \epsilon^{A_1} \{ H, \Omega^{(1)}_{A_1} \} \\ &= H(q,p) + \epsilon^{A_1} V^{B_1}_{A_1} \Omega^{(1)}_{B_1} \;, \end{aligned}$$

[5] In the following we treat time as discrete.

where ${V_A}^B$ has been defined in (5.13). The last term vanishes on the surface defined by the first class constraints. Furthermore

$$\begin{aligned}\sum_i \int_{t_i}^{t_f} dt\ p'_i \dot{q}'_i &= \sum_i \int_{t_i}^{t_f} dt\ (p_i + \delta p_i)(\dot{q}_i + \delta \dot{q}_i) \\ &\approx \sum_i \int_{t_i}^{t_f} dt\ p_i \dot{q}_i + \sum_i \int_{t_i}^{t_f} dt (\dot{q}_i \delta p_i - \dot{p}_i \delta q_i) \,.\end{aligned}$$

Making use of

$$\delta q_i = \frac{\partial G}{\partial p_i} \ ; \quad \delta p_i = -\frac{\partial G}{\partial q_i}$$

we have that

$$\dot{q}_i \delta p_i - \dot{p}_i \delta q_i = -\frac{dG}{dt} + \dot{\epsilon}^{A_1}(t) \Omega^{(1)}_{A_1} \ ,$$

so that

$$\sum_i \int_{t_i}^{t_f} dt\ p'_i \dot{q}'_i = \sum_i \int_{t_i}^{t_f} dt\ p_i \dot{q}_i + \int_{t_i}^{t_f} dt\ \dot{\epsilon}^{A_1}(t) \Omega^{(1)}_{A_1} + \left[p_i \frac{\partial G}{\partial p_i} - G \right]_{t_i}^{t_f} \ , \quad (10.25)$$

a result that was already obtained in chapter 5 (see (5.16)). The term involving $\dot{\epsilon}^{A_1}$ does not contribute to the functional integral because of the delta function $\delta[\Omega^{(1)}_{A_1}]$. The last term does also not contribute for variations vanishing at the end points of integration t_i and t_f, i.e., when $\epsilon^{A_1}(t_i) = \epsilon^{A_1}(t_f) = 0$. Making use of (10.22), (10.24) and (10.25), we are led from (10.20) to (10.21). This proves our assertion that the partition function does not depend on the choice of gauge. Correlators of non-gauge invariant quantities are of course gauge dependent.

Chapter 11

Dynamical Gauges. BFV Functional Quantization

11.1 Introduction

The path-integral quantization in so-called "non-dynamical" gauges, as discussed in the previous chapter, is not particularly suited for perturbative calculations. This is well known, and is intimately linked to the fact that the commutators of fields in such gauges have in general a non-local Dirac bracket structure. From the perturbative point of view one would like to deal with a formalism involving standard canonical commutation relations. One way to achieve this would be to go to a reduced phase space as discussed in the last section of chapter 4. In general this can however be done only at the expense of a non-local action. On the other hand, locality of the action and a canonical structure can be achieved by suitably enlarging the phase space by introducing extra degrees of freedom. For QED and QCD this is well known to particle physicists.

Quantization of gauge theories requires one to fix the gauge. Physical observables are insensitive to this since they are required to be invariant under all the local transformations which leave the Lagrangian invariant. Examples of such observables are the Maxwell field-strength tensor as well as non-local fermion bilinears including a Schwinger line integral. In this chapter we will be particularly interested in covariant gauges. Having fixed a gauge, an observable will no longer exhibit a local symmetry. What is then the criterion for identifying it as an observable? As we shall see, an observable will be required to be invariant under transformations generated by a nilpotent charge carrying ghost number one, the so-called BRST charge.

Faddeev and Popov [Fadeev 1967] were the first ones to give a prescription for QCD of how such a quantization can be carried out in configuration space. This prescription is known as the "Faddeev-Popov" trick. In QCD, for example, it leads to a new, gauge fixed effective Lagrangian involving dynamical ghosts, which exhibits a remnant of the original local symmetry, generated by a nilpotent charge. This symmetry was discovered by Becchi, Rouet and Stora, and independently by Tyutin [Becchi 1976], and will be referred to simply as BRST symmetry.

Although the Faddeev-Popov procedure is familiar to most theoretical particle physicists, we will discuss it in terms of a simple quantum mechanical example, in order to emphasize those general features which are characteristic for the quantization of any gauge system in a dynamical gauge. We first construct the effective gauge fixed Lagrangian L_{gf} following the method of Faddeev-Popov, and then proceed from here to obtain the corresponding gauge fixed Hamiltonian, whose structure we eventually compare with the "unitarizing" Hamiltonian H_U constructed along the lines discussed by Batalin, Fradkin, Fradkina and Vilkovisky in a series of sophisticated papers [Fradkin 1975,77/78, Batalin 1977/83b,c/86], from here on referred to as BFV.

In the following we shall denote by H any Hamiltonian weakly equivalent to the canonical Hamiltonian. When discussing examples we shall be more specific and denote, as always, the canonical Hamiltonian defined on the primary constrained surface by H_0.

11.2 Grassmann variables

Since in this and the following chapter we shall work extensively with (anticommuting) Grassmann variables, we begin this chapter with a brief review of Grassmann algebras and a discussion of the relevant expressions which will be needed in the remaining part of this book.

i) Grassmann algebra

The elements $\eta_1, ..., \eta_N$ are said to be the generators of a Grassmann algebra, if they anticommute among each other, i.e. if

$$\eta_i\eta_j + \eta_j\eta_i = 0, \quad i, j = 1, ..., N. \tag{11.1}$$

From here it follows that

$$\eta_i^2 = 0. \tag{11.2}$$

A general element of a Grassmann algebra is defined as a power series in the η_i's. Because of (11.2), however, this power series has only a finite number of terms:

$$f(\eta) = f_0 + \sum_i f_i\eta_i + \sum_{i\neq j} f_{ij}\eta_i\eta_j + ... + f_{12...N}\eta_1\eta_2...\eta_N \,. \tag{11.3}$$

As an example consider the function

$$g(\eta) = e^{-\sum_{i,j=1}^N \eta_i A_{ij}\eta_j} .$$

It is defined by the usual power series expansion of the exponential. Since the terms appearing in the sum - being quadratic in the Grassmann variables - commute among each other, we can also write $g(\eta)$ as follows

$$g(\eta) = \prod_{i,j} e^{-\eta_i A_{ij}\eta_j} ,$$

or, making use of (11.2),

$$g(\eta) = \prod_{\substack{i,j=1\\ i\neq j}}^N (1 - \eta_i A_{ij}\eta_j).$$

Next we consider the following function of a set of $2N$-Grassmann variables which we denote by $\eta_1, \dots, \eta_N, \bar\eta_1, \dots, \bar\eta_N$:

$$h(\eta,\bar\eta) = e^{-\sum_{ij} \bar\eta_i A_{ij}\eta_j} .$$

Proceeding as above, we now have that

$$h(\eta,\bar\eta) = \prod_{i,j=1}^N (1 - \bar\eta_i A_{ij}\eta_j).$$

Notice that in contrast to previous cases, this expression also involves diagonal elements of A_{ij}.

ii) Graded derivatives

When dealing with functions $f(\eta)$ of Grassmann valued variables care must be taken of the anticommuting properties (11.1) when taking derivatives of f. In particular left and right derivatives are no longer the same. Suppose we want to differentiate $f(\eta)$ with respect to η_i. Then the rules are the following:

a) If $f(\eta)$ does not depend on η_i, then $\partial_{\eta_i} f(\eta) = 0$.

b) If $f(\eta)$ depends on η_i, then the left derivative $\partial/\partial\eta_i$ is performed by first bringing the variable η_i (which never appears twice in a product!) all the way to the left, using the anticommutation relations (11.1), and then applying the rule

$$\frac{\partial^{(l)}}{\partial\eta_i}\eta_i = 1.$$

c) Correspondingly, we obtain the right derivative $\frac{\partial^{(r)}}{\partial\eta_i}$ by bringing the variable η_i all the way to the right and then applying the rule

$$\frac{\partial^{(r)}}{\partial\eta_i}\eta_i = 1\,.$$

Thus for example

$$\frac{\partial^{(l)}}{\partial\eta_i}\eta_j\eta_i = -\eta_j \quad (i \neq j),$$

or

$$\frac{\partial^{(r)}}{\partial\bar{\eta}_i}\bar{\eta}_i\eta_j = -\eta_j\,.$$

It follows from here, that the left- and right derivatives of a Grassmann odd operator F are the same, whereas there is a relative minus sign in the case of a Grassmann even operator B: [1]

$$\begin{aligned}\frac{\partial^{(l)}}{\partial\eta}F(\eta) &= \frac{\partial^{(r)}}{\partial\eta}F(\eta)\,,\\ \frac{\partial^{(l)}}{\partial\eta}B(\eta) &= -\frac{\partial^{(r)}}{\partial\eta}B(\eta)\,.\end{aligned}$$

Another property which can be easily verified, is that

$$\left\{\frac{\partial}{\partial\eta_i}, \frac{\partial}{\partial\eta_j}\right\} f(\eta) = 0\,,$$

where the derivatives are either left or right derivatives. Below we will present some formulae involving the derivatives of functions depending on "bosonic" (i.e. commuting) and "fermionic" (i.e. anticommuting) variables. To this effect we introduce the Grassmann signature of a function f:

$$\begin{aligned}\epsilon_f &= 0 \quad (bosonic)\\ \epsilon_f &= -1 \quad (fermionic)\end{aligned}$$

[1] F (B) stands symbolically for "fermionic" ("bosonic").

Sometimes we will also use the notation $\epsilon_f = \epsilon(f)$. Let us furthermore denote by θ the collection of bosonic variables $\{q_i\}$ and fermionic variables $\{\eta_\alpha\}$ on which a general function F depends. Thus

$$F(\theta) = F_0(q) + \sum_n \sum_{\{\ell_k\}} F^{(n)}_{\ell_1 \ldots \ell_n}(q)\eta_{\ell_1}\eta_{\ell_2}\ldots\eta_{\ell_n} \ .$$

Note that the nth term in the sum has Grassmann signature $\epsilon = 0$ if n is even, and 1, if n is odd. It is now easy to verify that the following differentiation rules hold:

$$\begin{aligned}
\frac{\partial^{(l)}}{\partial\theta^\ell}(fg) &= \frac{\partial^{(l)} f}{\partial\theta} g + (-1)^{\epsilon_f\epsilon_\theta} f \frac{\partial^{(l)} g}{\partial\theta} \ , \\
\frac{\partial^{(r)}}{\partial\theta^\ell}(fg) &= f\frac{\partial^{(r)} g}{\partial\theta^\ell} + (-1)^{\epsilon_g\epsilon_\theta} \frac{\partial^{(r)} f}{\partial\theta^\ell} g \ , \\
\frac{\partial^{(l)}}{\partial\theta^\ell}F(f(\theta)) &= \frac{\partial^{(l)} f}{\partial\theta^\ell}\frac{\partial^{(l)} F}{\partial f} \ , \\
\frac{\partial^{(r)}}{\partial\theta^\ell}F(f(\theta)) &= \frac{\partial^{(r)} F}{\partial f}\frac{\partial^{(r)} f}{\partial\theta} \ .
\end{aligned}$$

Let us apply these rules to some cases of interest. Consider the function

$$E(\bar\rho) = e^{\sum_j \bar\rho_j\eta_j},$$

where $\{\eta_i, \bar\rho_i\}$ are the generators of a Grassmann algebra. If they were ordinary c-numbers then we would have that

$$\frac{\partial^{(l)}}{\partial\bar\rho_i}E(\bar\rho) = \eta_i E(\bar\rho)\,.$$

This result is in fact also correct in general. To see this let us write $E(\bar\rho)$ in the form

$$E(\bar\rho) = \prod_j (1 + \bar\rho_j\eta_j).$$

Applying the rules of Grassmann differentiation, we have that

$$\frac{\partial^{(l)}}{\partial\bar\rho_i}E(\bar\rho) = \eta_i \prod_{j\neq i}(1 + \bar\rho_j\eta_j).$$

But because of the appearance of the factor η_i we are now free to include the extra term $1 + \bar\rho_i\eta_i$ in the above product. Hence we arrive at the above-mentioned naive result. It should, however, be noted, that the order of the

Grassmann variables in $\sum_i \bar{\rho}_i \eta_i$ was important. By reversing this order we get a minus sign, and the rule is not the usual one! By a similar argument one finds that

$$\frac{\partial^{(r)}}{\partial \rho_i} e^{\sum_j \bar{\eta}_j \rho_j} = \bar{\eta}_i e^{\sum_j \bar{\eta}_j \rho_j}.$$

iii) Integration over Grassmann variables

We now state the Grassmann rules for calculating integrals of the form

$$\int \prod_{i=1}^{N} d\eta_i f(\eta),$$

where $f(\eta)$ is a function whose general structure is given by (11.3). Since a given Grassmann variable can at most appear to the first power in $f(\eta)$, the following rules suffice to calculate an arbitrary integral [Berezin 1966]:

$$\int d\eta_i = 0, \quad \int d\eta_i \eta_i = 1.$$

When computing multiple integrals one must further take into account that the integration measures $\{d\eta_i\}$ also anticommute among themselves, as well as with all η_j's

$$[d\eta_i, d\eta_j]_+ = [d\eta_i, \eta_j]_+ = 0, \quad \forall\,, i, j\,. \tag{11.4}$$

From above we infer that

$$\int d\eta_i f(\eta) = \frac{\partial^{(l)}}{\partial \eta_i} f(\eta).$$

Hence the integral coincides with the left derivative.

These integration rules look very strange. But they are the appropriate ones to allow one to obtain the path integral representation of the generating function for gauge theories in a fixed gauge with a local effective action.

As an example let us apply these rules to the calculation of the following integral:

$$I = \int \prod_{\ell=1}^{N} d\bar{\eta}_\ell d\eta_\ell e^{-\sum_{i,j=1}^{N} \bar{\eta}_i A_{ij} \eta_j}. \tag{11.5}$$

We could have also denoted the Grassmann variables by $\eta_1, \ldots, \eta_{2N}$, by setting $\eta_{N+i} = \bar{\eta}_i$. But for reasons which will become clear later, we prefer the above notation. To evaluate (11.5), we first write the integrand in the form

$$e^{-\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j} = \prod_{i=1}^{N} e^{-\bar{\eta}_i \sum_{j=1}^{N} A_{ij} \eta_j}.$$

Since $\bar\eta_i^2 = 0$, only the first two terms in the expansion of the exponential will contribute. Hence

$$e^{-\sum_{i,j}\bar\eta_i A_{ij}\eta_j} = (1-\bar\eta_1 A_{1i_1}\eta_{i_1})(1-\bar\eta_2 A_{2i_2}\eta_{i_2})\dots(1-\bar\eta_N A_{Ni_N}\eta_{i_N}), \quad (11.6)$$

where on the rhs a summation over repeated indices i_ℓ ($\ell = 1,\cdots,N$) is understood. Now, because of the Grassmann integration rules (11.4), the integrand of (11.5) must involve the product of all the Grassmann variables. We therefore only need to consider the term

$$K(\eta,\bar\eta) = \sum_{i_1,\dots,i_N} \eta_{i_1}\bar\eta_1\eta_{i_2}\bar\eta_2\dots\eta_{i_N}\bar\eta_N A_{1i_1}A_{2i_2}\dots A_{Ni_N}, \quad (11.7)$$

where we have set $\bar\eta_k\eta_{i_k} = -\eta_{i_k}\bar\eta_k$ to eliminate the minus signs appearing in (11.6). The summation clearly includes only those terms for which all the indices $i_1,\dots,i_N$ are different. Now, the product of Grassmann variables in (11.7) is antisymmetric under the exchange of any pair of indices i_ℓ and $i_{\ell'}$. Hence we can write the above expression in the form

$$K(\eta,\bar\eta) = \eta_1\bar\eta_1\eta_2\bar\eta_2\dots\eta_N\bar\eta_N \sum_{i_1\dots i_N} \epsilon_{i_1 i_2\dots i_N} A_{1i_1}A_{2i_2}\dots A_{Ni_N},$$

where $\epsilon_{i_1 i_2\dots i_N}$ is the totally antisymmetric Levy-Civita tensor in N dimensions. Recalling the standard formula for the determinant of a matrix A, we therefore find that

$$K(\eta,\bar\eta) = (\det A)\eta_1\bar\eta_1\eta_2\bar\eta_2\dots\eta_N\bar\eta_N.$$

We now replace the exponential in (11.5) by this expression and obtain

$$I = \left[\prod_{i=1}^N \int d\bar\eta_i d\eta_i \eta_i\bar\eta_i\right]\det A = \det A.$$

Let us summarize our result for later convenience:

$$\int D(\bar\eta\eta)e^{-\sum_{i,j=1}^N \bar\eta_i A_{ij}\eta_j} = \det A, \quad D(\bar\eta\eta) = \prod_{\ell=1}^N d\bar\eta_\ell d\eta_\ell. \quad (11.8)$$

For completeness sake we mention another important formula used to calculate Green functions, although we will not need it in the following. This formula allows one to calculate integrals of the type

$$I_{i_1\cdots i_\ell i'_1\cdots i'_\ell} = \int D(\bar\eta\eta)\eta_{i_1}...\eta_{i_\ell}\bar\eta_{i'_1}\dots\bar\eta_{i'_\ell}e^{-\sum_{i,j=1}^N\bar\eta_i A_{ij}\eta_j}.$$

Consider the following generating functional

$$Z[\rho, \bar{\rho}] = \int D(\bar{\eta}\eta) e^{-\sum_{i,j} \bar{\eta}_i A_{ij} \eta_j + \sum_i (\bar{\eta}_i \rho_i + \bar{\rho}_i \eta_i)} , \tag{11.9}$$

where all indices are understood to run from 1 to N, and where the "sources" $\{\rho_i\}$ and $\{\bar{\rho}_i\}$ are now also anticommuting elements of the Grassmann algebra generated by $\{\eta_i, \bar{\eta}_i, \rho_i, \bar{\rho}_i\}$. To evaluate (11.9) we first rewrite the integral as follows:

$$Z[\rho, \bar{\rho}] = \left[\int D(\bar{\eta}\eta) e^{-\sum_{i,j} \bar{\eta}'_i A_{ij} \eta'_j} \right] e^{\sum_{i,j} \bar{\rho}_i A^{-1}_{ij} \rho_i}$$

where

$$\eta'_i = \eta_i - \sum_k A^{-1}_{ik} \rho_k, \quad \bar{\eta}'_i = \bar{\eta}_i - \sum_k \bar{\rho}_k A^{-1}_{ki},$$

and A^{-1} is the inverse of the matrix A. Making use of the invariance of the integration measure under the above transformation, [2] and of (11.8), one finds that

$$Z[\rho, \bar{\rho}] = \det A \, e^{\sum_{i,j} \bar{\rho}_i A^{-1}_{ij} \rho_j} .$$

Notice that in contrast to the bosonic case, this generating functional is proportional to $\det A$ (instead of $(\det A)^{-1/2}$).

After this brief mathematical digression, we now turn to a simple quantum mechanical analog of a theory with a local symmetry, quantized in a covariant gauge.

11.3 BFV quantization of a quantum mechanical model

The purpose of this section is to elucidate in terms of a simple model the main ideas going into the BFV quantization in configuration space of a theory with a local symmetry, and to arrive at a phase space representation of the partition function having the structure discussed in [Fradkin 1975/77, Batalin 1977].

Consider again the Lagrangian (3.52), i.e

$$L = \frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x - y)^2 . \tag{11.10}$$

The absence of the velocity $\dot{y}$ implies the primary constraint

$$\phi \equiv p_y = 0 . \tag{11.11}$$

[2] This is ensured by the Grassmann integration rules.

The associated canonical Hamiltonian (evaluated on the primary surface) is given by

$$H_0 = \frac{1}{2}p_x^2 - yp_x + \frac{1}{2}y^2 - \frac{1}{2}(x-y)^2 \tag{11.12}$$

and leads to a secondary constraint [3]

$$T \equiv x - p_x = 0 \ . \tag{11.13}$$

Rewriting H_0 in the form

$$H_0 = \frac{1}{2}(p_x^2 - x^2) + yT \ , \tag{11.14}$$

we see that the variable y plays the role of a Lagrange multiplier, multiplying the *secondary* constraint.

11.3.1 The gauge-fixed effective Lagrangian

The constraints (11.11) and (11.13) are evidently first class, thus signalizing that the Lagrangian (11.10) exhibits a local symmetry. Indeed the corresponding action is invariant under the infinitesimal local transformations

$$\delta x = \varepsilon(t) \ , \quad \delta y = \varepsilon(t) - \dot{\varepsilon}(t) \ . \tag{11.15}$$

In fact, it is also invariant under the finite transformation $x \to x + \alpha(t)$, $y \to y + \alpha(t) - \dot{\alpha}(t)$, with $\alpha(t)$ an arbitrary function. We now wish to implement on quantum level the following (dynamical) gauge condition, involving the time derivative of the "Lagrange multiplier" y:

$$\dot{y} + \chi(x, y) = 0 \ . \tag{11.16}$$

This gauge condition is analogous to the Lorentz condition in the Yang-Mills case. We implement this gauge condition following the procedure of Faddeev-Popov, which is well known in QCD. A priori this procedure is however just a prescription (Faddeev-Popov trick), since it departs from a divergent configuration space partition function, but which is known to work e.g. for QED or QCD. Nevertheless we shall use this prescription in order to motivate the axiomatic BFV formulation discussed in section 6.

[3] Following Batalin and Fradkin, we denote here first class secondary constraints by T_a.

In the following we now implement the gauge condition (11.16) á la Faddeev and Popov [Faddeev 1967]. We first define the following functional:

$$\Delta^{-1}[x;y] = \int D\alpha \prod_t \delta\left(\frac{d\,{}^{\alpha}\!y(t)}{dt} + \chi({}^{\alpha}\!x(t), {}^{\alpha}\!y(t))\right) \ , \qquad (11.17)$$

where Dz stands generically for the product $\prod_t dz(t)$, and ${}^{\alpha}\!z$ for the gauge transform of z with the gauge function $\alpha(t)$. Clearly $\Delta[x,y]$ is gauge invariant, i.e.

$$\Delta[{}^{\alpha}\!x, {}^{\alpha}\!y] = \Delta[x,y] \, .$$

The square brackets in $\Delta[x,y]$ reminds us that Δ is a functional of $x(t)$ and $y(t)$, i.e., that it depends on the time history of x and y. Next we introduce the unit element

$$1 = \Delta[x,y] \int D\alpha \prod_t \delta\left(\frac{d\,{}^{\alpha}\!y(t)}{dt} + \chi({}^{\alpha}\!x(t), {}^{\alpha}\!y(t))\right)$$

into the "naive" partition function $Z = \int DxDy\ exp(i \int dtL)$. Making use of the gauge invariance of $\Delta[x,y]$, the classical action, and the integration measure, i.e.

$$S[{}^{\alpha}\!x, {}^{\alpha}\!y] = S[x,y] \ , \quad D{}^{\alpha}\!xD{}^{\alpha}\!y = DxDy \ ,$$

one is led to the following expression for the partition function,

$$Z = \Omega \int DxDy\ \Delta[x,y] \prod_t \delta\Big(\dot{y}(t) + \chi(x(t), y(t))\Big)\ e^{iS} \ , \qquad (11.18)$$

where Ω is the infinite "gauge group volume" $\Omega = \int D\alpha$, S is the classical action associated with the Lagrangian (11.10), $S = \int dt\ L$, and where we have redefined the fields along the way. This relabeling does not affect gauge invariant correlation function. [4]

Because of the δ-function in (11.18) we only need to know $\Delta[x,y]$ for y satisfying the condition (11.16). This implies that the integration over α involves only the infinitesimal neighbourhood of $\alpha = 0$. Hence we have, using (11.15),

$$\Delta^{-1}[x,y] = \int \prod_t d\varepsilon(t)\ \prod_{t,t'} \delta\left(\int dt'\ D(t,t')\varepsilon(t')\right) = (detD)^{-1} \, ,$$

where D is the matrix [5]

$$D(t,t') = \left[\frac{d}{dt} - \frac{d^2}{dt^2} + \left(\frac{\partial\chi}{\partial x}\right) + \left(\frac{\partial\chi}{\partial y}\right)\left(1 - \frac{d}{dt}\right)\right]\delta(t-t') \, .$$

[4] We shall frequently refer to $x(t), y(t)$, etc. as "fields", since they are the one-dimensional analog of the fields in the case of non-denumerable degrees of freedom.

[5] We have suppressed the x, y arguments in $D(t,t')$.

By introducing the Grassmann valued Faddeev-Popov ghosts $c(t)$ and antighosts $\bar{c}(t)$ with ghost numbers $gh(c) = 1$, $gh(\bar{c}) = -1$, and recalling (11.8), one can write $\det D$ in the form [6]

$$detD = \Delta[x,y] = \int D\bar{c}Dc \; e^{i\int dt \; \tilde{L}_{gh}} \; ,$$

where

$$\tilde{L}_{gh} = \bar{c}(\dot{c} - \ddot{c}) + \bar{c}\frac{\partial\chi}{\partial x}c + \bar{c}\frac{\partial\chi}{\partial y}(c - \dot{c}) \; .$$

Finally, writing the product of the δ-functions in (11.18) as a Fourier transform

$$\prod_t \delta\Big(\dot{y} + \chi(x,y)\Big) = \int D\xi \; e^{i\int dt \;\; \xi(\dot{y}+\chi)} \; ,$$

and dropping the irrelevant (infinite) factor Ω, we arrive at the following form for the partition function

$$Z = \int DxDyD\xi D\bar{c}Dc \; e^{i\int dt \;\; L_{gf}} \; , \tag{11.19}$$

where

$$L_{gf}(x,y,\dot{x},\dot{y},\xi) = L(x,y,\dot{x}) + \xi(\dot{y} + \chi(x,y)) + \tilde{L}_{gh} \tag{11.20}$$

is the gauge fixed Lagrangian. For later purposes it will be convenient to write the ghost Lagrangian $\tilde{L}_{gh}$ in a form exhibiting only first order time derivatives, by dropping an irrelevant total time derivative:

$$\tilde{L}_{gh} \to L_{gh} = \dot{\bar{c}}(\dot{c} - c) + \bar{c}\frac{\partial\chi}{\partial x}c + \bar{c}\frac{\partial\chi}{\partial y}(c - \dot{c})\, . \tag{11.21}$$

Having fixed the gauge, the coordinate y has been promoted to a dynamical variable at the cost of a (t-dependent) Lagrange multiplier ξ. But we have lost of course the original gauge symmetry. Nevertheless, the action S_{gf} does exhibit a so-called BRST symmetry in the extended space, which is a remnant of the original local symmetry. It is not difficult to obtain the corresponding infinitesimal transformations. Let us write the BRST variation of any quantity F in the form [Baulieu 1986]

$$\delta_{\mathcal{B}} F = \epsilon s F \; , \tag{11.22}$$

[6] We are free to introduce the (conventional) factor i in the exponent, since it amounts to a change of variable $\bar{c} \to i\bar{c}$, and hence to a multiplication of the partition function by a complex number. This does not affect normalized correlation functions.

where ϵ is a global (constant) infinitesimal Grassmann valued parameter, and s is a Grassmann odd operator. Noting that $c^2 = 0$ and $(c - \dot{c})^2 = 0$, the transformations read

$$\begin{aligned} &sx = c\ , \quad sy = c - \dot{c}\ , \\ &sc = 0\ , \quad s\bar{c} = -\xi\ , \\ &s\xi = 0\ . \end{aligned} \tag{11.23}$$

The operator s acts from the *left* according to the graded product rule,

$$s(fg) = (sf)g + (-1)^{\epsilon(f)} f(sg) \tag{11.24}$$

where $\epsilon(f)$ is the Grassmann signature of f:

$$\begin{aligned} \epsilon(f) = 0: &\quad (f\ bosonic) \\ \epsilon(f) = 1: &\quad (f\ fermionic) \end{aligned}$$

An alternative way of writing the BRST variation (11.22) is

$$\delta_{\mathcal{B}} F = \hat{s} F \epsilon\ , \tag{11.25}$$

where the operator $\hat{s}$ now acts from the *right* according to the rule,

$$\hat{s}(fg) = f(\hat{s}g) + (-1)^{\epsilon(g)}(\hat{s}f)g\ . \tag{11.26}$$

The form (11.25) is encountered frequently in the literature. Note that the BRST variations $\delta_{\mathcal{B}} x$ and $\delta_{\mathcal{B}} y$ are obtained from (11.15) by simply replacing $\varepsilon(t)$ by $\epsilon c(t)$.

A very important feature of s is that it is a nilpotent operator, i.e.,

$$s^2 = 0\,. \tag{11.27}$$

In fact, this is a central feature of BRST transformations in general. A simple computation yields

$$\delta_{\mathcal{B}} L_{gf} = sL_{gf} = \epsilon \frac{d}{dt}(xc + \xi(c - \dot{c})). \tag{11.28}$$

Thus $\delta_B L_{gf}$ is just a total derivative, so that the action is invariant under the transformations (11.23), as was claimed above.

Furthermore, one readily checks, taking account of the Grassmann nature of s, that L_{gf} can be written in the form

$$L_{gf} = L - s\left[\bar{c}(\dot{y} + \chi(x, y))\right] - \frac{d}{dt}(\bar{c}(c - \dot{c}))\ . \tag{11.29}$$

Hence the corresponding gauge fixed Lagrangian differs (up to a total derivative) from the classical Lagrangian (11.10) by a BRST exact term, i.e. a term that can be written in the form $s\mathcal{F}$, and whose BRST variation vanishes *identically* by construction.

The equations of motion follow from the requirement that

$$0 = \delta S_{gf} = \int dt\ \delta L_{gf} = \sum_\alpha \int dt\ \left(\delta q_\alpha \frac{\partial^{(l)} L_{gf}}{\partial q_\alpha} + \delta \dot{q}_\alpha \frac{\partial^{(l)} L_{gf}}{\partial \dot{q}_\alpha} \right) \ ,$$

where δ now denotes an arbitrary variation, and q_α stands for any of the variables $x, y, c, \bar{c}$. Notice the appearance of the "left partial derivatives" (labeled by a superscript l), with the differentials dq_α and $d\dot{q}_\alpha$ appearing to the left of the derivatives. This ordering is of course only relevant for the anticommuting ghost variables. A similar expression holds with the differentials appearing to the right of the corresponding "right derivatives". The vanishing of the above variation implies the Euler-Lagrange equations of motion,

$$\frac{d}{dt}\left(\frac{\partial^{(l)} L_{gf}}{\partial \dot{q}_\alpha} \right) - \frac{\partial^{(l)} L_{gf}}{\partial q_\alpha} = 0 \ , \tag{11.30}$$

or their equivalent with right derivatives. In particular, for the ghosts we are led to the equations

$$\begin{aligned} &\ddot{c} - \dot{c} - \frac{\partial \chi}{\partial x} c - \frac{\partial \chi}{\partial y}(c - \dot{c}) = 0 \,, \\ &\ddot{\bar{c}} + \dot{\bar{c}} - \bar{c}\frac{\partial \chi}{\partial x} - \bar{c}\frac{\partial \chi}{\partial y} - \frac{d}{dt}\left(\bar{c}\frac{\partial \chi}{\partial y} \right) = 0 \,. \end{aligned}$$

11.3.2 The conserved BRST charge in configuration space

With the BRST symmetry of the action there is associated a conserved BRST charge. To obtain an expression for this charge we construct the corresponding conserved Noether current in the standard way by computing the *on-shell* BRST variation of the Lagrangian in two alternative ways. Consider first an off-shell variation. It is given by (11.28). On the other hand, making explicit use of the Euler-Lagrange equations of motion (11.30), one finds that the on-shell BRST variation has the form

$$\begin{aligned} \delta_{\mathcal{B}} L_{gf} &= \frac{d}{dt}\left(\delta_{\mathcal{B}} x \frac{\partial L_{gf}}{\partial \dot{x}} + \delta_{\mathcal{B}} y \frac{\partial L_{gf}}{\partial \dot{y}} + \delta_{\mathcal{B}} c \frac{\partial^{(l)} L_{gf}}{\partial \dot{c}} + \delta_{\mathcal{B}} \bar{c} \frac{\partial^{(l)} L_{gf}}{\partial \dot{\bar{c}}} \right) \\ &= \epsilon \frac{d}{dt}[(\dot{x} + y)c - 2\xi(\dot{c} - c)] \ . \end{aligned} \tag{11.31}$$

Taking the difference between (11.28) and (11.31) one obtains on shell

$$\frac{d}{dt}Q_{\mathcal{B}} = 0 \ , \quad (\textit{on-shell})$$

where

$$Q_{\mathcal{B}} = xc - (\dot{x} + y)c - \xi(c - \dot{c}) \ . \tag{11.32}$$

$Q_{\mathcal{B}}$ is called the BRST charge, which - when expressed in terms of phase space variables - will be the generator of BRST transformations in phase-space, as we shall see below.

11.3.3 The gauge fixed effective Hamiltonian

Our next objective is to obtain a phase-space representation of the partition function. Since our quantum mechanical model is to exemplify a generic structure of the Hamiltonian and phase-space representation of the partition function, we shall be rather pedantic in our analysis. To obtain the phase-space equivalent of (11.19), we first construct a "gauge fixed" Hamiltonian H_{gf} as the Legendre transform of L_{gf}. Since L_{gf} involves also the Faddeev-Popov ghosts, we must be careful in the ordering of Grassmann variables. In the following we carry out a standard analysis, keeping close watch on the ordering of factors.

Let $L(q, \dot{q})$ be the Lagrangian in question (in the present case $L \to L_{gf}$). Its differential is given by

$$dL = \sum_{\alpha} \left(dq_{\alpha} \frac{\partial^{(l)} L}{\partial q_{\alpha}} + d\dot{q}_{\alpha} \frac{\partial^{(l)} L}{\partial \dot{q}_{\alpha}} \right) \ .$$

A corresponding statement holds where the differentials are standing to the right of the right partial derivatives. The q_{α}'s stand for any of the c-number and Grassmann valued variables. Note that the ordering of the products is important. Correspondingly we introduce canonical momenta conjugate to the coordinates as left derivatives with respect to the velocities:

$$p_{\alpha} = \frac{\partial^{(l)} L}{\partial \dot{q}_{\alpha}} \ .$$

Then

$$dL = \sum_{\alpha} \left(dq_{\alpha} \frac{\partial^{(l)} L}{\partial q_{\alpha}} + d\dot{q}_{\alpha} p_{\alpha} \right) \ .$$

Writing $d\dot{q}_\alpha p_\alpha = -\dot{q}_\alpha dp_\alpha + d(\dot{q}_\alpha p_\alpha)$, [7] and making use of the Euler-Lagrange-equations of motion (11.30), one is led to the differential

$$dH = \sum_\alpha (\dot{q}_\alpha dp_\alpha - dq_\alpha \dot{p}_\alpha)\,, \tag{11.33}$$

where

$$H = \sum_\alpha (\dot{q}_\alpha p_\alpha - L)$$

is the Hamiltonian. Note again the ordering of the velocities and momenta. This ordering is correlated with the definition of the canonical momenta which are defined here in terms of left derivatives. If the q_α and p_α were all independent (unconstrained) variables, then the Hamilton equations of motion could be read off immediately from (11.33). This is however not so in the case under consideration, since (11.20) describes a constrained system with primary second class constraints $\xi - p_y = 0$ and $p_\xi = 0$. These primary constraints must be implemented in a functional integral for the partition function either explicitly, or by introducing Lagrange multipliers as new integration variables.

Denote the momenta conjugate to c and $\bar{c}$ by [8]

$$\bar{P} = \frac{\partial^{(l)} L}{\partial \dot{c}}\,, \quad P = \frac{\partial^{(l)} L}{\partial \dot{\bar{c}}}\,. \tag{11.34}$$

Hence the ghost number of P ($\bar{P}$) is opposite to that of $\bar{c}$ (c); i.e. $gh(P) = -gh(\bar{P}) = 1$. For our model we then have

$$\begin{aligned} p_x &= \dot{x} + y\,, \quad p_y = \xi\,, \quad p_\xi = 0\,, \\ \bar{P} &= -\dot{\bar{c}} + \bar{c}\frac{\partial\chi}{\partial y}\,, \quad P = \dot{c} - c\,. \end{aligned} \tag{11.35}$$

Implementing the primary second class constraints $p_y - \xi = 0$, $p_\xi = 0$, we obtain for the corresponding partition function

$$Z = \int DxDp_x \int DyDp_y \int DcD\bar{P} \int D\bar{c}DP\ e^{iS_{gf}}\,, \tag{11.36}$$

where

$$S_{gf} = \int dt\ \left(\dot{x}p_x + \dot{y}p_y + \dot{c}\bar{P} + \dot{\bar{c}}P - H_{gf}\right)\,, \tag{11.37}$$

[7] The differential of a product is still given by $d(AB) = (dA)B + A(dB)$, also if one is dealing with Grassmann variables.

[8] We follow here the conventions of [Henneaux/Teitelboim, 1992].

with H_{gf} given by,

$$H_{gf} = \left[\dot{x}p_x + \dot{y}p_y + \dot{\xi}p_\xi + \dot{c}\bar{P} + \dot{\bar{c}}P - L_{gf}\right]_{(p_\xi=0,\xi=p_y)} ,$$

or

$$H_{gf} = H_0 - \bar{P}(c+P) - \bar{c}\left[\frac{\partial\chi}{\partial x}c - \frac{\partial\chi}{\partial y}P\right] - p_y\chi \, . \tag{11.38}$$

Here H_0 is the canonical Hamiltonian (11.14) associated with the classical Lagrangian (11.10), evaluated on the associated primary constrained surface $p_y = 0$. It is now an easy matter to show that upon performing the integrations over p_x, $\bar{P}$ and P in (11.36) one recovers (11.19) with p_y playing the role of ξ.

Now comes an important point. Since we have integrated out the (second class) primary constraints, the partition function takes the form of an unconstrained system with the integrand expressed in terms of independent phase space variables. Hence from (11.33) the equations of motion corresponding to a stationary action take the form [9]

$$\dot{q}_\alpha = \frac{\partial^{(r)}H_{gf}}{\partial p_\alpha} \; , \quad \dot{p}_\alpha = -\frac{\partial^{(l)}H_{gf}}{\partial q_\alpha} \; . \tag{11.39}$$

From (11.23) and (11.35) we obtain the BRST transformation laws in phase space,

$$\begin{aligned} sx &= c \; , \quad sy = -P \, , \quad s\xi = 0 \; , \\ sc &= 0 \; , \quad s\bar{c} = -p_y \\ sp_x &= c \; , \quad sp_y = 0 \, , \quad sp_\xi = 0 \\ sP &= 0 \; , \quad s\bar{P} = -T \; , \end{aligned} \tag{11.40}$$

where T is given by (11.13). [10] Note again that s is nilpotent, i.e., $s^2 = 0$. Recalling (11.24) one readily checks that the gauge fixed Hamiltonian is BRST

[9] Note the appearance of the right derivative, as follows from (11.33) with $H \to H_{gf}$ and the prescribed ordering of canonical variables (which also include the ghosts).

[10] In deriving the transformation law $s\bar{P} = -T$ one must make use of the equation of motion. From (11.35) and (11.23) we have

$$s\bar{P} = s\left(-\dot{\bar{c}} + \bar{c}\frac{\partial\chi}{\partial y}\right) = \dot{p}_y - (p_y + \bar{c}s)\,\frac{\partial\chi}{\partial y},$$

and from the equation of motion for p_y we obtain

$$\dot{p}_y = -\frac{\partial^{(l)}H_{gf}}{\partial y} = -T + \bar{c}\left(\frac{\partial^2\chi}{\partial x\partial y}c - \frac{\partial^2\chi}{\partial y^2}P\right) + p_y\frac{\partial\chi}{\partial y} = -T + \bar{c}s\frac{\partial\chi}{\partial y} + p_y\frac{\partial\chi}{\partial y} \; .$$

Hence $s\bar{P} = -T$.

invariant:

$$sH_{gf} \equiv 0\,.$$

11.3.4 The BRST charge in phase space

We next construct the conserved BRST charge $Q_{\mathcal{B}}$, which implements the BRST transformations in phase space. From (11.32) and (11.35) we infer that

$$Q_{\mathcal{B}} = p_y P + Tc\,. \tag{11.41}$$

The Hamilton equations of motion (11.39) and transformation laws (11.40) can also be written in Poisson bracket form, by introducing generalized Poisson brackets. We demand that the properties of these Poisson brackets should be those of a Berezin algebra [Berezin 1966]

$$\begin{aligned}\{F,G\} &= -(-1)^{\epsilon(F)\epsilon(G)}\{G,F\}\ ,\\ \{FG,H\} &= F\{G,H\} + (-1)^{\epsilon(G)\epsilon(H)}\{F,H\}G\ .\end{aligned} \tag{11.42}$$

Furthermore the Hamilton equations of motion (11.39) should take the standard form

$$\dot{q}_i = \{q_i, H_{gf}\}\ ,\quad \dot{p}_i = \{p_i, H_{gf}\}\ ,$$

and the product rule (11.24) should be implemented correctly by the BRST charge in terms of Poisson brackets. One is then led to the following definition of the generalized (graded) Poisson bracket:

$$\{F,G\} = \sum_k (-1)^{\epsilon(Q_k)}\left(\frac{\partial^{(r)}F}{\partial Q_k}\frac{\partial^{(l)}G}{\partial P_k} - (-1)^{\epsilon(F)\epsilon(G)}\frac{\partial^{(r)}G}{\partial Q_k}\frac{\partial^{(l)}F}{\partial P_k}\right)\ , \tag{11.43}$$

or equivalently

$$\{F,G\} = \sum_k (-1)^{\epsilon(Q_k)}\left(\frac{\partial^{(r)}F}{\partial Q_k}\frac{\partial^{(l)}G}{\partial P_k} - (-1)^{\epsilon(Q_k)\epsilon(P_k)}\frac{\partial^{(r)}F}{\partial P_k}\frac{\partial^{(l)}G}{\partial Q_k}\right)\ , \tag{11.44}$$

where (Q_k, P_k) denote canonically conjugate pairs. There are several ways of rewriting this generalized Poisson bracket. In the above form the transition to the commutator of operators is most transparent. Hence in particular we have for the non-vanishing Poisson brackets of the (Grassmann valued) ghosts, [11]

$$\{\bar{P}, c\} = \{P, \bar{c}\} = -1\ . \tag{11.45}$$

[11] It is common to write the Poisson brackets in this form in order to align them with the usual Poisson brackets of commuting variables of Grassman parity zero: $\{p,x\} = -1$. For Dirac fermions described by the Lagrangian density $\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi$ one usually has

$$\{\psi_\alpha(x), \psi^\dagger_\beta(y)\} = -i\delta_{\alpha\beta}\delta(\vec{x}-\vec{y})\ ,$$

In the following $\{\ ,\ \}$ will always denote the generalized (graded) Poisson bracket. With this definition, one now readily verifies, that $Q_{\mathcal{B}}$ induces the above BRST transformations according to

$$s\mathcal{F} = \{Q_{\mathcal{B}}, \mathcal{F}\}\ , \tag{11.46}$$

where $\mathcal{F}$ is a function of the independent variables $x, y, p_x, p_y, c, \bar{c}, P, \bar{P}$. According to (11.22) this corresponds to

$$\delta_{\mathcal{B}}\mathcal{F} = \epsilon\{Q_{\mathcal{B}}, \mathcal{F}\}$$

with ϵ to the *left* of the Poisson bracket. Alternatively we have (cf. (11.25), where $\hat{s}$ was introduced),

$$\hat{s}\mathcal{F} = \{\mathcal{F}, Q_{\mathcal{B}}\}\ , \tag{11.47}$$

where $\hat{s}$ acts from the right, or

$$\delta_{\mathcal{B}}\mathcal{F} = \{\mathcal{F}, Q_{\mathcal{B}}\}\epsilon\,.$$

Furthermore, one verifies that $Q_{\mathcal{B}}$ is conserved:

$$\frac{dQ_{\mathcal{B}}}{dt} = \{Q_{\mathcal{B}}, H_{gf}\} = 0\,.$$

Expressed in terms of $Q_{\mathcal{B}}$ the nilpotency of s takes the form

$$\{Q_{\mathcal{B}}, Q_{\mathcal{B}}\} = 0\,. \tag{11.48}$$

This is a nonempty statement since $Q_{\mathcal{B}}$ is a Grassmann odd operator. Indeed, from

$$0 = s^2F = s(sF) = s\{Q_{\mathcal{B}}, F\} = \{Q_{\mathcal{B}}, \{Q_{\mathcal{B}}, F\}\}$$

one finds, upon making use of the Jacobi identity

$$(-1)^{\epsilon_F\epsilon_H}\{F, \{G, H\}\} + cycl.\ perm. = 0$$

that

$$\{F, \{Q_{\mathcal{B}}, Q_{\mathcal{B}}\}\} = 0\,.$$

Since this expression vanishes for all F we are led to the conclusion (11.48).

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which corresponds to defining the momenta via the right derivative:

$$\pi_\alpha = \frac{\partial^{(r)}\mathcal{L}}{\partial\psi_\alpha} = i\psi_\alpha^\dagger\,.$$

If the momenta were defined in terms of left derivatives (as was done above), we would have $\pi_\alpha = -i\psi_\alpha^\dagger$, and the Poisson bracket would be of the form (11.45), that is, $\{\pi_\alpha(x), \psi_\beta(y)\} = -\delta_{\alpha\beta}\delta(\vec{x} - \vec{y})$.

Let us now write the action (11.37) and BRST charge in a compact form by collecting the Grassmann variables with ghost number +1 into the following vector [Fradkin 1975/77],

$$\eta^A := \begin{pmatrix} P \\ c \end{pmatrix} .$$

Correspondingly we define the vector $\vec{\bar{\mathcal{P}}}$ with components canonically conjugate to η^A, carrying ghost number -1,

$$\bar{\mathcal{P}}_A := \begin{pmatrix} \bar{c} \\ \bar{P} \end{pmatrix} .$$

According to (11.45) their Poisson bracket is given by

$$\{\eta^A, \bar{\mathcal{P}}_B\} = -\delta^A_B . \tag{11.49}$$

In this notation, the gauge fixed action (11.37) takes the form

$$S_{gf} = \int dt \ \left(\dot{q}^i p_i + \dot{\eta}^A \bar{\mathcal{P}}_A - H_{gf}\right) , \tag{11.50}$$

where we have dropped a surface term, and where $q^i := (q_x, q_y)$ and $p_i := (p_x, p_y)$. Furthermore the BRST charge (11.41) becomes

$$Q_{\mathcal{B}} = \eta^A G_A ,$$

where

$$G_A := (\phi, T) \tag{11.51}$$

are the primary and secondary first class constraints (11.11) and (11.13), associated with the *classical* Lagrangian (11.10).

As we shall see in the following section, $Q_{\mathcal{B}}$ will in general involve additional higher ghost contributions. The appearance of $G_A \eta^A$ as the leading term of $Q_{\mathcal{B}}$ is quite natural, since with (11.46) it implements the classical gauge transformations (11.15) with $\epsilon(t)$ replaced by $c(t)$. For the quantum mechanical model this leading term is actually the only one. This feature is a consequence of the fact that for this model the constraints are in strong involution, i.e., $\{G_A, G_B\} = 0$.

Next we rewrite the gauge fixed Hamiltonian H_{gf} in the form proposed by Fradkin, Vilkovisky and Batalin [Fradkin 1977, Batalin 1977]. It is instructive to do this in two different ways.

a) Version 1

Let H_0 be the canonical Hamiltonian (11.12) of the classical theory evaluated on the primary constrained surface $p_y = 0$. The gauge fixed Hamiltonian is given by (11.38). Consider the Poisson algebra of the constraints (11.51). Being a first class system this algebra is closed and reads [12]

$$\{G_A, G_B\} = U_{AB}^C G_C = 0 \ , \tag{11.52}$$

i.e., all the coefficients U_{AB}^C vanish in the present case. Consider further the Poisson algebra of the Hamiltonian H_0 with the constraints, which we write in the form

$$\{H_0, G_A\} = V_A^B G_B \ . \tag{11.53}$$

For the case in question, the only non-vanishing coefficients V_A^B are given by

$$V_1^2 = V_2^2 = 1 \ .$$

One readily checks that the gauge fixed Hamiltonian (11.38) can be written in the form

$$H_{gf} = H_{\mathcal{B}} + \{\Psi, Q_{\mathcal{B}}\} \ ,$$

where $H_{\mathcal{B}}$ is the "BRST" Hamiltonian

$$H_{\mathcal{B}} = H_0 + \eta^A V_A^B \bar{\mathcal{P}}_B = H_0 + (P + c)\bar{P} \ , \tag{11.54}$$

which is BRST invariant, and

$$\Psi = \bar{\mathcal{P}}_A \psi^A \ , \quad \psi^A := (\chi, 0) \ .$$

The BRST invariance of (11.54) can be easily verified by making use of (11.49), the definition of ${V_A}^B$ in (11.53), and of (11.46). One finds that

$$sH_{\mathcal{B}} = \{Q_{\mathcal{B}}, H_{\mathcal{B}}\} = 0 \ .$$

The Grassmann valued function Ψ with ghost number -1 is referred to in the literature as the "fermion gauge-fixing function". The Poisson bracket $\{\Psi, Q_{\mathcal{B}}\}$ is a so-called "BRST exact term", whose BRST variation vanishes identically for any Ψ because of the nilpotency of $Q_{\mathcal{B}}$, i.e. (11.48).

b) Version 2

The above decomposition of H_{gf} into a BRST invariant contribution $H_{\mathcal{B}}$ and a BRST exact term is not unique. Below we consider an alternative way of decomposing the gauge fixed Hamiltonian.

Let H_0' be the Hamiltonian obtained from (11.14) by omitting the term yT, i.e., the term involving the "Lagrange multiplier" y multiplying the secondary

[12] Our conventions for the definition of the structure functions follow most common practice among physicists, and differ from those of Fradkin and Vilkovisky.

constraint. This Hamiltonian also satisfies an algebra with the constraints which is of the form,

$$\{H_0', G_A\} = V_A{}'^B G_B \,,$$

where the only non-vanishing coefficient $V_A'{}^B$ is given by $V_2'{}^2 = 1$. The algebra of the constraints remains of course the same as before. One then readily verifies that H_{gf} can again be written in the form,

$$H_{gf} = H_{\mathcal{B}}' + \{\Psi', Q_{\mathcal{B}}\} \,,$$

where

$$\Psi' = \bar{\mathcal{P}}_A \psi'^A \,, \qquad \psi'^A := (\chi, -y)$$

and

$$H_{\mathcal{B}}' = H_0' + \eta^A V_A'^B \bar{\mathcal{P}}_B = H_0' + c\bar{P} \,.$$

Note that now the "Lagrange multiplier" y appears in the gauge condition through the fermion gauge-fixing function Ψ ! Note also that the above expressions for the BRST Hamiltonian $H_{\mathcal{B}}'$ and gauge fixed Hamiltonian H_{gf} are both BRST invariant, as can be readily verified by making use of (11.40):

$$sH_{\mathcal{B}}' = \{Q_{\mathcal{B}}, H_B'\} \equiv 0 \,.$$

Furthermore, $s\{\Psi', Q_{\mathcal{B}}\} \equiv 0$ because of the generalized Jacobi identity,

$$\begin{aligned}(-1)^{\epsilon_{F_1}\epsilon_{F_3}}\{\{F_1, F_2\}, F_3\} + (-1)^{\epsilon_{F_2}\epsilon_{F_1}}\{\{F_2, F_3\}, F_1\}& \\ + (-1)^{\epsilon_{F_3}\epsilon_{F_2}}\{\{F_3, F_1\}, F_2\} = 0 \,,& \qquad (11.55)\end{aligned}$$

and the nilpotency of s (or equivalently (11.48)). Hence $H_{\mathcal{B}}'$ and H_{gf} are both BRST invariant. But also the action (11.50) is invariant, since the kinetic contribution to S_{gf} is by itself BRST invariant. Indeed, let

$$\xi^\rho := (q^i, \eta^A) \,, \quad \omega_\rho := (p_i, \bar{\mathcal{P}}_A) \,.$$

The kinetic contributing to the action density is then given by $\dot{\xi}^\rho \omega_\rho$. Under an infinitesimal BRST transformation

$$\begin{aligned}\xi^\rho \to \xi'^\rho &= \xi^\rho + \epsilon\{Q_{\mathcal{B}}, \xi^\rho\} \,, \\ \omega_\rho \to \omega_\rho' &= \omega_\rho + \epsilon\{Q_{\mathcal{B}}, \omega_\rho\} \,, \qquad (11.56)\end{aligned}$$

the kinetic contribution transforms infinitesimally as follows

$$\dot{\xi}'^\rho \omega_\rho' = \dot{\xi}^\rho \omega_\rho + \epsilon\Big(\frac{d}{dt}\{Q_{\mathcal{B}}, \xi^\rho\}\Big)\omega_\rho + \dot{\xi}^\rho \epsilon\{Q_{\mathcal{B}}, \omega_\rho\} + O(\epsilon^2) \,.$$

Now, from (11.43) we have

$$\{Q_{\mathcal{B}}, \xi^\rho\} = -\frac{\partial Q_{\mathcal{B}}}{\partial \omega_\rho} \ ,$$
$$\{Q_{\mathcal{B}}, \omega_\rho\} = (-1)^{\epsilon(\xi_\rho)} \frac{\partial Q_{\mathcal{B}}}{\partial \xi^\rho} \ ,$$

where we have dropped the "left-right" labels on the partial derivatives, since $Q_{\mathcal{B}}$ is a Grassmann odd operator. One then finds, upon making use of the Grassmann nature of ϵ to eliminate the phase factor, that

$$\dot{\xi}'^\rho \omega'_\rho = \dot{\xi}^\rho \omega_\rho + \epsilon \frac{d}{dt}\left(Q_{\mathcal{B}} - \frac{\partial Q_{\mathcal{B}}}{\partial \omega_\rho}\omega_\rho\right) \ . \tag{11.57}$$

Hence the action S_{gf} is BRST invariant provided that

$$\left(Q_{\mathcal{B}} - \frac{\partial Q_{\mathcal{B}}}{\partial \omega_\rho}\omega_\rho\right)_{t=-\infty}^{t=\infty} = 0 \,. \tag{11.58}$$

Finally we note, that since the transformation (11.56) is canonical, the integration measure in (11.36) is formally also BRST invariant. Hence we conclude that the partition function is (formally) invariant under BRST transformations. [13]

The above two versions of the quantum mechanical example illustrate in pedantic detail some general features of the BRST formalism for quantizing theories with a local gauge symmetry, which will be formulated axiomatically in section 5. Before doing so it is instructive to carry out a similar analysis for the SU(3) Yang-Mills theory in the Lorentz gauge. The so-called "alpha"-gauges will be dealt with in appendix B.

11.4 Quantization of Yang-Mills theory in the Lorentz gauge

As in the case of our quantum mechanical example, the BRST quantization of the $SU(3)$ Yang-Mills gauge theory can be carried out in configuration - or phase space. We begin again with a discussion of the configuration space approach, following the steps taken in our quantum mechanical model of the previous section, and generalizing them to a system with an infinite number of degrees of freedom.

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[13] As we shall see in the following chapter, this invariance may be broken by gauge anomalies in the case of an infinite number of degrees of freedom, such as in a Quantum Field Theory.

Let us first summarize some of the well known facts. Our starting point will be the Yang-Mills Lagrangian density

$$\mathcal{L} = -\frac{1}{2} tr F_{\mu\nu} F^{\mu\nu} \ , \tag{11.59}$$

with

$$F_{\mu\nu} = t_a F^a_{\mu\nu} \ ,$$

where

$$[t_a, t_b] = i f_{abc} t_c \ , \quad \mathrm{tr}(t_a t_b) = \frac{1}{2}\delta_{ab} \ .$$

Here "tr" denotes the trace with respect to the internal symmetry group $\mathcal{G}$, t_a are the generators of the group (which for $SU(3)$ are given conventionally by the 3×3 Gell-Mann matrices $\frac{1}{2}\lambda_a$ $(a = 1, \ldots, 8)$), and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \tag{11.60}$$

is the field-strength tensor, with f_{abc} the structure constants of the group $\mathcal{G}$. The Yang-Mills action is invariant under the local (gauge) transformations

$$A_\mu \to {}^G\!A_\mu = G A_\mu G^{-1} + \frac{i}{g} G \partial_\mu G^{-1} \ , \tag{11.61}$$

where $G(x)$ is an element of the gauge group,

$$G(x) = e^{i\Lambda(x)} \ , \tag{11.62}$$

with

$$A_\mu = A^a_\mu t_a \ , \quad \Lambda = \Lambda^a t_a \ .$$

From (11.59) we have for the momenta conjugate to A^μ,

$$\pi^a_0 = 0 \, , \quad \pi^a_i = F^a_{i0} \, .$$

The first set of equations represent primary constraints, which we denote by ϕ_a:

$$\phi_a \equiv \pi^a_0 = 0 \ . \tag{11.63}$$

The canonical Hamiltonian evaluated on the subspace defined by the *primary* constraints is given by

$$H_0 = \int d^3x \ [\frac{1}{2} \sum_{i,a} (\pi^a_i)^2 + \frac{1}{4} \sum_{i,j,a} (F^a_{ij} F^a_{ij})] - \sum_{i,a,b} \int d^3x \ A^a_0 \mathcal{D}^i_{ab} \pi^b_i \, , \tag{11.64}$$

where [14]

[14] For ease of reading the colour index will appear sometimes as subscript or superscript. It will then be understood that repeated indices are always summed, unless otherwise stated.

$$\mathcal{D}^i_{ab} = \delta_{ab}\partial^i - g f_{abc} A^i_c \tag{11.65}$$

is the covariant derivative. Persistence of the primary constraints leads to secondary constraints, i.e., Gauss's law,

$$T_a \equiv \mathcal{D}^i_{ab}\pi^b_i = 0\,. \tag{11.66}$$

The secondary constraints can be shown to satisfy the (equal time) Poisson algebra:

$$\{T_a(x), T_b(y)\} = g f_{abc} T_c(x)\delta(\vec{x}-\vec{y}) \ . \tag{11.67}$$

All the remaining Poisson brackets of the primary and secondary constraints vanish. Collecting the constraints into a vector

$$G_A := (\vec{\phi}, \vec{T}) \tag{11.68}$$

we have that

$$\{G_A(x), G_B(y)\} = \tilde{U}^C_{AB} G_C(x)\delta(\vec{x}-\vec{y}) \ , \tag{11.69}$$

with

$$\tilde{U}^C_{AB} := \begin{pmatrix} \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & g f_{abc} \end{pmatrix} \ . \tag{11.70}$$

Furthermore

$$\{H_0, G_A(x)\} = \tilde{V}^B_A(x) G_B(x) \ ,$$

where [15]

$$\tilde{V}^B_A(x) = -\begin{pmatrix} \underset{\sim}{0} & \delta_{ab} \\ \underset{\sim}{0} & g f_{abc} A^c_0(x) \end{pmatrix} \ . \tag{11.72}$$

Note that in this theory the number of secondary constraints equals the number of primary constraints, and that each primary constraint generates one gauge identity. Furthermore all the constraints are first class.

i) Gauge fixed effective Lagrangian

Although the following steps are just an extension to an infinite number of degrees of freedom of the steps followed in our quantum mechanical example, and

[15] After some algebra one finds that

$$\{H'_0, T_a\} = (\mathcal{D}^i\mathcal{D}^j)_{ab} F^b_{ij} = \frac{1}{2}[\mathcal{D}^i, \mathcal{D}^j]_{ab} F^b_{ij} \ , \tag{11.71}$$

where H'_0 is given by the first integral in (11.64), and $\mathcal{D}^i$ is the matrix defined in (11.65). But

$$[\mathcal{D}_i, \mathcal{D}_j]_{ab} = -g f_{abc} F^c_{ij} \ .$$

Hence the rhs of (11.71) vanishes.

can be found in any textbook on Quantum Field Theory, we shall nevertheless present them for the readers convenience.

Consider the naive functional integral

$$Z = \int DA^\mu e^{-i\int d^4x\ \frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a} . \tag{11.73}$$

Since we wish to implement the Lorentz gauge $\partial^\mu A_\mu^a = 0$, we define, following Faddeev and Popov [Faddeev 1967], the functional $\Delta[A]$ via

$$\Delta[A] \int DG \prod_{x,a} \delta(\partial^\mu\ {}^G\!A_\mu(x)) = 1\ , \tag{11.74}$$

where DG denotes the gauge invariant *Haar*-measure of the symmetry group in question. It then follows that $\Delta[A]$ is also gauge invariant, i.e.

$$\Delta[A] = \Delta[{}^G\!A]\ . \tag{11.75}$$

Introducing the identity (11.74) into the functional integral (11.73), and making use of the property (11.75), as well as of the gauge invariance of the integration measure and Lagrangian density,

$$DA = D\ {}^G\!A\ , \quad \mathcal{L}({}^G\!A) = \mathcal{L}(A)\ ,$$

we can rewrite the partition function as follows:

$$Z = \Omega_{\mathcal{G}} \int DA\ \Delta[A] \prod_{x,a} \delta(\partial^\mu A_\mu^a(x)) e^{i\int d^4x \mathcal{L}(A)}\ , \tag{11.76}$$

where $\Omega_{\mathcal{G}}$ is the group volume $\Omega_{\mathcal{G}} = \int \mathcal{D}G$, and where we have renamed the integration variables along the way. This renaming does not change gauge invariant correlation functions, since they do not depend on the gauge functions Λ^a in (11.62). From (11.76) it follows that we only need to compute $\Delta[A]$ for configurations satisfying the gauge conditions $\partial^\mu A_\mu^a = 0$. In this case the integral (11.74) will only receive contributions from gauge transformations in the infinitesimal neighbourhood of the identity,

$$\delta A_a^\mu = \mathcal{D}_{ab}^\mu \Lambda^b\ , \tag{11.77}$$

where $\mathcal{D}_{ab}^\mu$ is the covariant derivative

$$\mathcal{D}_{ab}^\mu = \delta_{ab}\partial^\mu - g f_{abc} A_c^\mu\ . \tag{11.78}$$

Hence for gauge fields satisfying $\partial^\mu A_\mu^a(x) = 0$, we have (see e.g. [Itzykson 1980])

$$\Delta^{-1}[A] = \int D\Lambda \prod_{a,x} \delta\left(\int d^4x'\ M_{ab}(x,x')\Lambda^b(x')\right)\ ,$$

where

$$M_{ab}(x,x') = \partial_\mu \mathcal{D}^\mu_{ab}(A(x))\delta^4(x-x') \ ,$$

or, making use of (11.8) (see footnote 6)

$$\begin{aligned}\Delta[A] &= \det(\partial^\mu \mathcal{D}_\mu) \\ &\approx \int D\bar{c}Dc \; e^{i\int d^4x \ \bar{c}_a \partial_\mu \mathcal{D}^\mu_{ab} c^b} \ ,\end{aligned}$$

with $c^a, \bar{c}_a$ Grassmann valued (Faddeev-Popov) ghost and antighost fields, to which we associate the ghost numbers

$$gh(c^a) = 1 \, , \quad gh(\bar{c}_a) = -1 \ .$$

Dividing out the group volume factor in (11.76), we are thus led to the following expression for the partition function

$$Z = \int DA \int D\bar{c}\ Dc \prod_{x,a} \delta(\partial^\mu A^a_\mu(x)) e^{i\int d^4x [\mathcal{L} + \bar{c}_a \partial_\mu \mathcal{D}^\mu_{ab} c^b]} \ . \tag{11.79}$$

Realizing the gauge condition as a Fourier transform via a Lagrange multiplier field, the so-called Nakanishi-Lautrup field B^a,

$$\prod_{x,a} \delta(\partial^\mu A^a_\mu(x)) = \int DB\, e^{i\int d^4x \ B_a(x)\partial^\mu A^a_\mu(x)} \ ,$$

one finally obtains

$$Z_{gf} = \int DB \int DAD\bar{c}Dc\, e^{i\int d^4x \mathcal{L}_{gf}} \ , \tag{11.80}$$

where $\mathcal{L}_{gf}$ is the "gauge fixed" Lagrangian

$$\mathcal{L}_{gf} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + B^a \partial_\mu A^\mu_a - (\partial_\mu \bar{c}_a)(\mathcal{D}^\mu_{ab} c^b) \ . \tag{11.81}$$

Here we have dropped an irrelevant four-divergence. The Euler–Lagrange equations, corresponding to stationary points of the action, read

$$\begin{gathered}\mathcal{D}^{ab}_\nu F^{\mu\nu}_b + \partial^\mu B^a - g f_{abc} (\partial^\mu \bar{c}_b) c^c = 0 \, , \\ \partial_\mu (\mathcal{D}^\mu_{ab} c^b) = 0 \, , \quad \mathcal{D}^\mu_{ab} (\partial_\mu \bar{c}_b) = 0 \, , \\ \partial^\mu A^a_\mu = 0 \ .\end{gathered} \tag{11.82}$$

Note that upon implementing the relativistic gauge, the Lagrange multiplier field A^a_0 of the original Lagrangian has been promoted to a dynamical field; i.e., the action involves a time derivative of A^a_0.

Although the gauge fixed Lagrangian (11.81) breaks the original gauge invariance (11.61), the action still possesses a global symmetry, as was first noticed by Becchi, Rouet and Stora, and also independently by Tyutin [Becchi 1976]. For A_μ^a the infinitesimal BRST transformation has the form (11.77), but with the arbitrary function $\Lambda^a(x)$ replaced by $\epsilon c^a(x)$, i.e.

$$\delta_{\mathcal{B}} A_a^\mu = \epsilon \mathcal{D}_{ab}^\mu c^b \ , \tag{11.83}$$

where ϵ is a global Grassmann valued parameter. The change in the gauge fixed action induced by these transformations is compensated by the following transformations of the ghost and Nakanishi-Lautrup fields:

$$\begin{aligned} \delta_{\mathcal{B}} \bar{c}_a &= -\epsilon B^a \ , \\ \delta_{\mathcal{B}} c^a &= -\epsilon \frac{g}{2} f_{abc} c^b c^c \ , \\ \delta_{\mathcal{B}} B^a &= 0 \,. \end{aligned}$$

The verification of the invariance of the action involves some straightforward algebra. It makes use of the Jacobi identity for the double commutators of the gauge generators, which leads to the well-known relation for the structure functions,

$$f_{abd} f_{dce} + f_{bcd} f_{dae} + f_{cad} f_{dbe} = 0 \ , \tag{11.84}$$

as well as to

$$\delta_{\mathcal{B}} F_{\mu\nu}^a = \epsilon g f_{abc} c^b F_{\mu\nu}^c \, ,$$

whose verification requires some algebra. In fact, one finds, making use of the Jacobi identity that the off-shell BRST variation of the Lagrangian density just contributes a (vanishing) surface term to the gauge-fixed action. Thus

$$\delta_{\mathcal{B}} \mathcal{L}_{gf} = \epsilon \partial_\mu (B^a \mathcal{D}_{ab}^\mu c^b) \ , \tag{11.85}$$

where use has been made of (11.84). As in the previous section, it is convenient in the following to free the BRST variation from the ϵ parameter by defining the operator s via (11.22).

The BRST symmetry of the action can be made manifest, once one realizes that, up to a four-divergence, $\mathcal{L}_{gf}$ can be expressed as the sum of the classical Lagrangian (11.59) and a BRST exact term, i.e. a term which can be written in the form $s\mathcal{F}$, whose BRST variation vanishes identically:

$$\mathcal{L}_{gf} = -\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} - s \left(\bar{c}_a \partial^\mu A_\mu^a \right) - \partial_\mu (\bar{c}_a \mathcal{D}_{ab}^\mu c^b) \,.$$

In mathematical terms it means that the gauged fixed *action* lies in the cohomology of the original (not gauge fixed) classical action, which is invariant under the restricted BRST transformations (11.83). This turns out to be also a general feature of quantum Lagrangians in a fixed gauge, the gauge fixing condition being hidden in a BRST exact term.

We next obtain the conserved BRST charge associated with the BRST symmetry. To this effect we construct the corresponding Noether current in the standard way by computing the on-shell variation of the Lagrangian density in two alternative ways, as already described in the previous section. The *off shell* BRST variation of $\mathcal{L}_{gf}$ is given by (11.85). On the other hand, the on-shell variation can be written in the form

$$\delta_B \mathcal{L}_{gf}|_{on-shell} = \epsilon\partial_\mu \left(sA_\nu^a \frac{\partial \mathcal{L}_{gf}}{\partial(\partial_\mu A_\nu^a)} + sc^a \frac{\partial^{(l)}\mathcal{L}_{gf}}{\partial(\partial_\mu c^a)} + s\bar{c}_a \frac{\partial^{(l)}\mathcal{L}_{gf}}{\partial(\partial_\mu \bar{c}_a)} \right) \ , \quad (11.86)$$

where, as before, $\partial^{(l)}$ denotes the "left derivative", and where use has been made of the equations of motion, which are given by an expression analogous to (11.30), i.e.,

$$\partial_\mu \frac{\partial^{(l)}\mathcal{L}_{gf}}{\partial(\partial_\mu \Theta_\alpha)} - \frac{\partial^{(l)}\mathcal{L}_{gf}}{\partial \Theta_\alpha} = 0 \ .$$

Here Θ_α stands generically for any of the fields $A_\mu^a, c^a, \bar{c}^a, B^a$. Expression (11.85) is valid off and on-shell. Going on-shell in (11.85), and taking the difference between (11.86) and (11.85), one arrives at the following (on-shell) expression for the conserved BRST current:

$$\mathcal{J}_{\mathcal{B}}^\mu = F_a^{\mu\nu}(\mathcal{D}_\nu)_{ab}c^b - B^a \mathcal{D}_{ab}^\mu c^b + \frac{g}{2} f_{abc}(\partial^\mu \bar{c}_a)c^b c^c \ .$$

The corresponding time-independent charge is therefore given by

$$Q_{\mathcal{B}} = \int d^3x \mathcal{J}_{\mathcal{B}}^0 = \int d^3x \ \left(c^a \mathcal{D}_{ab}^i F_{i0}^b - B^a \mathcal{D}_{ab}^0 c^b + \frac{g}{2} f_{abc}(\partial_0 \bar{c}_a)c^b c^c \right) \ . \quad (11.87)$$

$Q_{\mathcal{B}}$ is the BRST charge, which - when expressed in terms of phase space variables - becomes the generator of the BRST transformations in phase space, as we shall see further below.

ii) The gauge fixed Hamiltonian

Our aim is to obtain the phase-space version of the partition function (11.79). The procedure parallels closely the one described in the previous section. We first compute the gauge fixed Hamiltonian density via the Legendre transform of $\mathcal{L}_{gf}$. From (11.81) we obtain the canonical momenta conjugate to A_a^μ, c^a, $\bar{c}_a$ and B^a:

$$\pi_0^a = B^a \ , \quad \pi_B^a = 0 \ , \quad \pi_i^a = F_{i0}^a \ , \quad \bar{P}_a = \partial_0 \bar{c}_a \ , \quad P^a = -\mathcal{D}_{ab}^0 c^b \ . \qquad (11.88)$$

The first two equations are primary constraints. The corresponding gauge fixed Hamiltonian density is readily found to read (up to a spacial divergence) :

$$\mathcal{H}_{gf} = \mathcal{H}_0 - B^a \partial^i A_i^a + \mathcal{H}_{gh} ,$$

where $\mathcal{H}_{gh}$ is the ghost Hamiltonian density,

$$\mathcal{H}_{gh} = (\partial_i \bar{c}_a)\mathcal{D}^i_{ab} c^b - g f_{abc} \bar{P}_a c^b A_0^c + \bar{P}_a P^a ,$$

and $\mathcal{H}_0$ is given by

$$\mathcal{H}_0 = \frac{1}{2}\sum_{i,a}(\pi_i^a)^2 + \frac{1}{4}\sum_{i,j,a}(F_{ij}^a)^2 - \sum_{i,a,b} A_0^a \mathcal{D}^i_{ab}\pi_i^b .$$

Taking account of the primary constraints, the phase-space version of the quantum partition function now takes the form

$$Z_{gf} = \int D\mu \ \prod_{x,a}\delta(\pi_0^a(x) - B^a(x))\prod_{x,a}\delta(\pi_B^a(x)) \ e^{i\tilde{S}_{gf}} \ , \tag{11.89}$$

where

$$\tilde{S}_{gf} = \int d^4x \ \left(\dot{A}_a^\mu \pi_\mu^a + \dot{B}^a \pi_B^a + \dot{c}^a \bar{P}_a + \dot{\bar{c}}_a P^a - \mathcal{H}_{gf}\right) \ ,$$

and the integral is carried out over all variables on which the integrand depends; i.e, $D\mu = DADBD\pi D\pi_B D\bar{c}DcDPD\bar{P}$. Notice that, as in the case of our quantum mechanical example, the velocities always appear to the left of the corresponding conjugate momenta. Performing the integration in (11.89) over B^a and π_B^a the phase-space functional integral becomes

$$Z_{gf} = \int DAD\bar{c}Dc \int D\pi DPD\bar{P} \ e^{iS_{gf}} \ , \tag{11.90}$$

where S_{gf} is given by

$$S_{gf} = \int d^4x \ \left(\dot{A}_a^\mu \pi_\mu^a + \dot{c}^a \bar{P} + \dot{\bar{c}}_a P^a - \mathcal{H}_{gf}\right) \ ,$$

with

$$\mathcal{H}_{gf} = \mathcal{H}_0 - \pi_0^a \partial^i A_i^a - g f_{abc}\bar{P}_a c^b A_0^c - \bar{c}_a \partial_i \mathcal{D}^i_{ab} c^b + \bar{P}_a P^a \ . \tag{11.91}$$

It is now an easy matter to show that by integrating out the momenta one is led to the original functional integral in configuration space. Note that the partition function (11.90) has the canonical form characteristic of an unconstrained system, the "Lagrange multiplier" A_0^a having been promoted to a dynamical variable.

Consider now the BRST charge. From (11.87) and (11.88) we obtain

$$Q_{\mathcal{B}} = \int d^3x \left(P^a \phi_a + c^a T_a + \frac{g}{2} f_{abd} \bar{P}_a c^b c^d \right) \ ,$$

where ϕ_a and T_a are the primary and secondary constraints of the *original* theory defined by $\mathcal{L}$. They are given by (11.63) and (11.66), respectively. The action S_{gf} is invariant under the infinitesimal transformations generated by $Q_{\mathcal{B}}$, which can be written in the form of (11.46). $Q_{\mathcal{B}}$ has Grassmann signature $\epsilon(Q_{\mathcal{B}}) = 1$ and ghost number $gh(Q_{\mathcal{B}}) = 1$, and, as one can readily verify, is *nilpotent*, i.e., its graded Poisson bracket with itself vanishes. The BRST symmetry transformations induced by $Q_{\mathcal{B}}$ according to (11.46), read,

$$\begin{aligned}
&sA_a^0 = -P^a = \mathcal{D}^0_{ab} c^b \ , \quad sA_a^i = \mathcal{D}^i_{ab} c^b \ , \quad sB^a = 0 \ , \\
&sc^a = -\frac{g}{2} f_{abc} c^b c^c \ , \quad s\bar{c}^a = -\pi_0^a = -B^a \ , \\
&s\pi_0^a = 0 \ , \quad s\pi_i^a = -g f_{abc} c^b \pi_i^c \ , \quad s\pi_B^a = 0 \ , \\
&sP_a = 0 \ , \quad s\bar{P}_a = -T_a - g f_{abc} \bar{P}_b c^c \ .
\end{aligned}$$

In analogy to our quantum mechanical model we now introduce the following definitions, [16]

$$\eta^A(x) := \begin{pmatrix} \vec{P}(x) \\ \vec{c}(x) \end{pmatrix} ,$$

where $\vec{P}$ and $\vec{c}$ are n-component vectors, with n the number of first class primary constraints; i.e., $n = 8$ for $SU(3)$. Correspondingly we define the momenta canonically conjugate to the η^A's:

$$\bar{\mathcal{P}}_A(x) := \begin{pmatrix} \vec{\bar{c}}(x) \\ \vec{\bar{P}}(x) \end{pmatrix}$$

with Poisson brackets,

$$\{\eta^A(x), \bar{\mathcal{P}}_B(y)\} = -\delta^A_B \delta(\vec{x} - \vec{y}) \ .$$

Similar to the case of the quantum mechanical model, the gauge-fixed Hamiltonian density (11.91) can again be written in the compact form

$$\mathcal{H}_{gf} = \mathcal{H}_{\mathcal{B}} + \{\Psi, Q_{\mathcal{B}}\} \ ,$$

[16] The notation for the ghosts has been chosen such as to suggest that $c^a(x)$ and $\bar{c}_a(x)$ are the ghost and antighost fields that appear in the configuration space formulation of the partition function, as one is used to from textbooks on field theory, and that $P^a(x)$, $\bar{P}_a(x)$ are canonical momenta which are to be integrated over.

where

$$\mathcal{H}_\mathcal{B} = \mathcal{H}_0 + \eta^A \tilde{V}_A^B \bar{\mathcal{P}}_B$$

is BRST invariant by construction, V is the matrix with elements (11.72), and Ψ is given by

$$\Psi = \bar{\mathcal{P}}_A \psi^A\,, \quad \psi^A := (\{\partial^i A_i^a(x)\}, \vec{0})\,,$$

and carries ghost number -1.

Paralleling the quantum mechanical case, there is an alternative way of decomposing the gauge fixed Hamiltonian density in BRST invariant contributions. One easily verifies that $\mathcal{H}_{gf}$ can also be written in the form

$$\mathcal{H}_{gf} = \mathcal{H}'_\mathcal{B} + \{\Psi', Q_\mathcal{B}\}\ ,$$

where

$$\mathcal{H}'_\mathcal{B} = \mathcal{H}'_0 + \eta^A \tilde{V}_A^{\prime B} \bar{\mathcal{P}}_B$$

with $\mathcal{H}'_0$ the Hamiltonan density evaluated on the *full* constrained surface,

$$\mathcal{H}'_0 = \frac{1}{2}\sum_{i,a}(\pi_i^a)^2 + \frac{1}{4}\sum_{i,j,a}(F_{ij}^a)^2\ ,$$

and

$$\Psi' = \bar{\mathcal{P}}_A \psi'^A\,, \quad \psi'^A := (\{\partial^i A_i^a(x)\}, \{A_0^a(x)\})\ .$$

The coefficient functions $V_A^{\prime B}$ are now defined by

$$\{H'_0, G_A\} = \tilde{V}_A^{\prime B} G_B$$

and vanish in the present case. [17]

Finally, it is an easy exercise to verify that after integrating out the momenta, one is led to the Faddeev-Popov result (11.79). This does not yet prove that the Faddeev-Popov trick leads to the correct result for the partition function in the Lorentz gauge. To show this, we need a formulation which exhibits explicitly the freedom of choosing an arbitrary gauge. We shall discuss this point in section 6.

11.5 Axiomatic BRST approach

We now use the insight we have gained from our examples and take BRST invariance of the quantum Lagrangian and Hamiltonian as our fundamental

[17] See footnote 15.

principle for constructing the partition function. In the following we will assume that our system is purely first class.

The question we want to answer is: given a classical Lagrangian or Hamiltonian together with the constraints, what is the partition function describing the corresponding quantum theory? If one is dealing with a purely second class system, then the answer has been given in chapter 10. What about if one is dealing with a first class system, i.e., a system exhibiting a local (gauge) symmetry? What we know about such a system is that the first class constraints satisfy a closed Poisson algebra, and furthermore, that the Poisson bracket of the Hamiltonian with the constraints is a linear combination of the constraints. These properties, as well as the freedom in the choice of gauge, must manifest themselves in the quantum partition function. Our discussion in the last two sections, which was based on examples where the Faddeev-Popov trick is applicable, has shown that the invariance of the phase-space action under a global symmetry transformation generated by a nilpotent operator plays a central role in the construction of the quantum theory. In fact, the BRST invariant action contained a kinetic part involving *all* variables, including the ghosts and anti-ghosts, a BRST invariant Hamiltonian, whose construction only required the knowledge of the classical canonical Hamiltonian and its Poisson algebra with the constraints, and a BRST exact term implementing a gauge condition. These observations can be taken as basis for the axiomatic approach of Batalin, Fradkin and Vilkovisky [Fradkin 1975/77, Batalin 1977], which enables one to also quantize theories that cannot be handled by the Faddeev-Popov trick, in very general types of gauges. The proof of the correctness of such an axiomatic approach will be indirectly provided by the so-called "FV Principal Theorem" of Fradkin and Vilkovisky which states, that the BRST invariant partition function thus constructed is independent of the "fermion gauge-fixing function" [Fradkin 1975/77].

11.5.1 The BRST charge and Hamiltonian for rank one theories

In the following we shall assume that the number of (first class) secondary constraints equals the number of (first class) primaries, and that with each primary there is associated a gauge identity. [18] This is the case in particular for QED and $SU(N)$ gauge theories. Following the notation of the previous sections, we collect the constraints associated with a *classical* Lagrangian $L(\{q_i\},\{\dot{q}_i\})$ into a vector $\vec{G}$,

$$\{G_A\} = (\{\phi_a\},\{T_a\}) \ ,$$

[18] Although not stated explicitly by the authors, their formalism appears to be tuned to this case.

where $a = 1, 2, \cdots, n$, with n the number of primary constraints $\phi_a = 0$, and secondary constraints $T_a = 0$. The first class constraints and Hamiltonian satisfy an algebra of the form (11.52) and (11.53), where the structure functions U_{AB}^C and $V_A{}^B$ may depend on the coordinates and their conjugate momenta. As we have seen we can take for H_0 the canonical Hamiltonian evaluated either on Γ_P or on the full constrained surface Γ. Our aim is to construct the Hamiltonian $H_{\mathcal{B}}$ which is invariant under BRST transformations generated by a nilpotent operator, and subject to some gauge condition implemented via a BRST exact term. Hence one must first obtain an expression for this nilpotent operator.

i) The BRST charge

The BRST charge has Grassmann signature $\epsilon(Q_{\mathcal{B}}) = 1$, and carries ghost number $gh(Q_{\mathcal{B})}) = 1$. "Fields" which do not involve the ghosts should transform in the usual way, as determined from the classical action, but with the gauge degrees of freedom replaced by ghosts. These transformations leave the classical action invariant. The quantum mechanical model and the Yang-Mills theory studied in sections 3 and 4 suggest that in order to construct the BRST charge and Hamiltonian, one should introduce for each primary first class constraint $\phi_a = 0$ a ghost-antighost pair $(c^a, \bar{c}_a)$, as well as a corresponding canonically conjugate pair $(\bar{P}_a, P^a)$.

As in the previous sections, it turns out convenient to define the $2n$-component vector $\vec{\eta}$ carrying ghost number 1,

$$\eta^A := \begin{pmatrix} \vec{P} \\ \vec{c} \end{pmatrix} , \tag{11.92}$$

where $\vec{P}$ and $\vec{c}$ are n-component vectors. An arbitrary component of $\vec{\eta}$ will be labeled by a capital letter, i.e. η^A. On the other hand we reserve small latin letters for the components of $\vec{P}$ and $\vec{c}$, i.e., P^a and c^a. Corresponding to η^A we define the momentum $\bar{\mathcal{P}}_A$ canonically conjugate to η^A, i.e.,

$$\{\eta^A, \bar{\mathcal{P}}_B\} = -\delta_B^A , \tag{11.93}$$

where the (generalized) Poisson brackets have been defined in (11.43). The vector canonically conjugate to $\vec{\eta}$ carries ghost number -1 and has the form

$$\bar{\mathcal{P}}_A := \begin{pmatrix} \vec{\bar{c}} \\ \vec{\bar{P}} \end{pmatrix} , \tag{11.94}$$

where, as already mentioned, the components of $\vec{\bar{c}}$ and $\vec{\bar{P}}$ are labeled by a lower index, i.e., $\bar{c}_a$ and $\bar{P}_a$. Hence the number of canonically conjugate ghost pairs $(\eta^A, \bar{\mathcal{P}}_A)$ equals the number of first class constraints. Since the BRST charge carries Grassmann signature and ghost number 1, one is then led to the following ansatz,

$$Q_{\mathcal{B}} = Q^{(0)} + \sum_{n\geq 1} \sum_{A_1\cdots B_1\cdots} \mathcal{Q}_{A_1 A_2\cdots A_{n+1}}^{B_1 B_2\cdots B_n} \eta^{A_1} \cdots \eta^{A_{n+1}} \bar{\mathcal{P}}_{B_1} \cdots \bar{\mathcal{P}}_{B_n} \ , \qquad (11.95)$$

where

$$Q^{(0)} = \eta^A G_A \ , \qquad (11.96)$$

and where the tensors $\mathcal{Q}_{A_1 A_2\cdots A_{n+1}}^{B_1 B_2\cdots B_n}$ do not depend on the ghost variables. The order of the Polynomial in $\bar{\mathcal{P}}$ defines the "rank" of the BRST generator. For example, the abelian pure Maxwell theory and Yang-Mills theory have rank zero and one, respectively.

The appearance of $\eta^A G_A$ as the leading term of $Q_{\mathcal{B}}$ is required, since it implements the gauge transformations associated with the classical Lagrangian, where the independent c-number gauge parameters are replaced by Grassmann variables (ghosts). We will refer to (11.96) as the initial condition on $Q_{\mathcal{B}}$. For the quantum mechanical models discussed above the leading term in (11.95) is actually the only one. This is a consequence of the fact that for this system the constraints are in strong involution. On the other hand, the appearance of an additional term in the case of QCD is due to the fact that the Gauss law constraints are only in weak involution (cf. eq. (11.67)).

From the nilpotency requirement, $\{Q_{\mathcal{B}}, Q_{\mathcal{B}}\} = 0$, and the knowledge of the algebra of the first class constraints G_A, one can determine the coefficients $\mathcal{Q}_{A_1\ldots A_{n+1}}^{B_1\ldots B_n}$ in an iterative way. For many theories of interest, $\mathcal{Q}_{A_1\ldots A_{n+1}}^{B_1\ldots B_n}$ vanishes for $n > 1$, i.e. their rank is unity. In this case the expression for $Q_{\mathcal{B}}$ simplifies dramatically and we have the following theorem: [19]

Theorem 1

With the initial condition

$$Q^{(0)} = \eta^A G_A$$

and the involutive algebra [20]

$$\{G_A, G_B\} = U_{AB}^C G_C \ , \qquad (11.97)$$

[19] In the case of a field theory, the labels $A, B, \cdots$ will include also a corresponding spacial coordinate (unless the coordinate dependence is displayed explicitly), and sums over $A, B\cdots$ will include spacial integrations over corresponding coordinates.

[20] In contrast to [Fradkin 1975] we follow here the more familiar convention for defining the structure functions.

the BRST charge for a theory of rank one is given by [21]

$$Q_{\mathcal{B}} = \eta^A G_A + \frac{1}{2}\bar{\mathcal{P}}_C U^C_{AB}\eta^A\eta^B \, . \tag{11.98}$$

The above conditions are satisfied, in particular, for $SU(N)$ Yang-Mills theories. We leave the proof of this theorem for Appendix B.

ii) The gauged fixed BRST Hamiltonian

As we have already pointed out, the usual requirement of gauge invariance for an observable $\mathcal{O}$ is replaced within the gauge fixed formalism by the requirement of BRST invariance,

$$\{Q_{\mathcal{B}}, \mathcal{O}\} = 0 \, . \tag{11.99}$$

We are interested in particular in the case where $\mathcal{O}$ is the BRST-Hamiltonian $H_{\mathcal{B}}$. This Hamiltonian can only differ from the Hamiltonian associated with the classical theory (which is in involution with the first class constraints) by terms involving ghost variables. Furthermore it must have vanishing Grassmann signature and carry ghost number zero. We are thus led to the following Ansatz for the BRST Hamiltonian:

$$H_{\mathcal{B}} = H + \sum_{n\geq 1}\sum_{A_1\cdots B_1\cdots} \mathcal{H}^{B_1\ldots B_n}_{A_1\ldots A_n}\eta^{A_1}\ldots\eta^{A_n}\bar{\mathcal{P}}_{B_1}\ldots\bar{\mathcal{P}}_{B_n} \; ,$$

where the tensors $\mathcal{H}^{B_1\ldots B_n}_{A_1\ldots A_n}$ only depend on q and p. For the case where $Q_{\mathcal{B}}$ is of rank one, we then have the following theorem:

Theorem 2

For $Q_{\mathcal{B}}$ of rank one, and $H(q,p)$ satisfying the involutive algebra

$$\{H, G_A\} = V^B_A G_B \; , \tag{11.100}$$

with $V^B_A(q,p)$ subject to the condition that the completely antisymmetric part of $\{V^B_A, U^C_{DE}\}$ with respect to any pair of upper (lower) indices vanishes, the BRST Hamiltonian is given by

$$H_{\mathcal{B}} = H + \eta^A V^B_A \bar{\mathcal{P}}_B \; . \tag{11.101}$$

We leave the proof for Appendix C.

[21] This BRST charge is referred to as the *minimal* BRST charge (of rank one), in contrast to a *non-minimal* BRST charge constructed in an even larger space. For an example see section 7.

As we have seen in our examples, we can choose for H either H_0, the canonical Hamiltonian on the primary surface Γ_P, or H'_0, the canonical Hamiltonian on the fully constrained surface Γ.

iii) BRST Cohomology: The unitarizing Hamiltonian H_U

Once $H_{\mathcal{B}}$ has been constructed, the corresponding action

$$S_{\mathcal{B}} = \int dt \ \left(\dot{q}^i p_i + \dot{\eta}^A \bar{\mathcal{P}}_A - H_{\mathcal{B}}\right)$$

is BRST invariant. But this cannot be the entire story, since this action does not involve any gauge fixing term, but makes only use of the classical Hamiltonian, the algebraic properties of the constraints (11.97), and the involutive algebra (11.100). In fact, whereas the construction of the BRST charge via the ansatz (11.95) can be shown to be unique up to canonical transformations [Henneaux/Teitelboim 1992], the above construction of the BRST Hamiltonian is not unique. Thus we can always add to $H_{\mathcal{B}}$ a BRST exact term $\{\Psi, Q_{\mathcal{B}}\}$, with Ψ an arbitrary function with Grassmann signature $\epsilon(\Psi) = 1$ and ghost number $gh(\Psi) = -1$, without changing the conditions underlying its construction. It turns out that a particular choice of Ψ also implies a particular choice of gauge, as we have already seen in our examples. [22] For this reason Ψ is referred to in the literature as the "fermion gauge fixing function". The corresponding modified Hamiltonian is the so-called "unitarizing Hamiltonian" in the language of [Batalin 1977], and can be identified with the gauge-fixed Hamiltonian H_{gf} in our examples. We are thus led to the following BRST invariant and unitarizing Hamiltonian

$$H_U = H_{\mathcal{B}} + \{\Psi, Q_{\mathcal{B}}\} \ . \tag{11.102}$$

Note that H_U is just the extension of the BRST invariant Hamiltonian by a BRST exact term. It is quite remarkable that the corresponding quantum partition function is insensitive to different choices of Ψ. This is the content of a central theorem by Fradkin and Vilkovisky [Fradkin 1977]:

11.5.2 FV Principal Theorem

The path integral

$$Z_\Psi = \int DqDpD\eta D\bar{\mathcal{P}} e^{i\int_{t_1}^{t_2} dt\left(\dot{q}^\alpha p_\alpha + \dot{\eta}^A \bar{\mathcal{P}}_A - H_U(q,p,\eta,\bar{\mathcal{P}})\right)} \tag{11.103}$$

[22] Note that $\Psi = 0$ is also a possible choice.

with the "unitarizing" Hamiltonian

$$H_U = H_{\mathcal{B}} + \{\Psi, Q_{\mathcal{B}}\} \ , \tag{11.104}$$

$$\{Q_{\mathcal{B}}, H_{\mathcal{B}}\} = 0, \quad \{Q_{\mathcal{B}}, Q_{\mathcal{B}}\} = 0 \ ,$$

and with $Q_{\mathcal{B}}$ satisfying the boundary condition

$$\left[\sum_i p_i \frac{\partial Q_{\mathcal{B}}}{\partial p_i} + \sum_A \bar{\mathcal{P}}_A \frac{\partial Q_{\mathcal{B}}}{\partial \bar{\mathcal{P}}_A} - Q_{\mathcal{B}}\right]_{t_1}^{t_2} = 0$$

is independent of the gauge fixing function $\Psi = \Psi(q,p,\eta,\mathcal{P})$, carrying Grassmann signature $\epsilon(\Psi) = 1$ and ghost number $gh(\Psi) = -1$.

Note that this boundary condition is nothing but (11.58) for finite t_1 and t_2. It ensures the BRST invariance of the kinetic contribution to the action. Note also that the so-called "unitarizing Hamiltonian" corresponds to the gauge fixed Hamiltonian in our examples. We leave the proof of this important theorem for Appendix D. By imposing "inititial conditions" on the BRST charge and Hamiltonian, one makes contact with the gauge theory of interest.

From the FV theorem, and theorems 1 and 2, it follows in particular, that for theories of rank one the following statement holds:

Let G_A und H be an irreducible set of functions of the phase space variables $\{q_\alpha\}, \{p_\alpha\}$, with the involutive algebra

$$\begin{aligned} \{G_A, G_B\} &= U_{AB}^C G_C \ , \\ \{H, G_A\} &= V_A^B G_B \ , \end{aligned} \tag{11.105}$$

and with $V_A^B(q,p)$ subject to the condition that the completely antisymmetric part of $\{V_A^B, U_{DE}^C\}$ with respect to any pair of upper (lower) indices vanishes. Then the following choice of $H_{\mathcal{B}}$ and $Q_{\mathcal{B}}$

$$\begin{aligned} H_{\mathcal{B}} &= H + \eta^A V_A^B \bar{\mathcal{P}}_B \ , \\ Q_{\mathcal{B}} &= G_A \eta^A + \frac{1}{2}\bar{\mathcal{P}}_C U_{AB}^C \eta^A \eta^B \end{aligned} \tag{11.106}$$

ensures the Ψ independence of the functional integral (11.103).

The FV principle theorem provides us with the structure of the partition function which ensures that Z_Ψ is actually independent of Ψ. This does not ensure however that it is the correct quantum partition function associated with a classical Lagrangian, characterized by an involutive algebra (11.105). For this to be the case one must show that there exists a choice for Ψ such that BRST invariant correlators of operators satisfying (11.99) map onto corresponding gauge invariant correlators in a physical gauge where the correct path integral representation is known. This cannot be shown in general, but must be checked in each case separately. Before demonstrating this for the case of the $SU(3)$ Yang-Mills theory, let us however first obtain a more explicit expression for the unitarizing Hamiltonian for theories of rank *one* and "fermion gauge fixing functions" Ψ linear in the ghosts.

11.5.3 A large class of gauges

For fermion gauge fixing functions linear in the ghosts, the general structure is fixed by recalling that Ψ carries ghost number $gh(\Psi) = -1$:

$$\Psi = \bar{\mathcal{P}}_A \psi^A(q,p) \ .$$

Our choice of Ψ in our quantum mechanical and Yang-Mills examples fall into this category. With this choice we have

$$\{\Psi, Q_{\mathcal{B}}\} = \bar{\mathcal{P}}_A\{\psi^A, Q_{\mathcal{B}}\} + \psi^A\{\bar{\mathcal{P}}_A, Q_{\mathcal{B}}\} \ .$$

For a theory of rank one (see (11.106))

$$\{\psi^A, Q_{\mathcal{B}}\} = \{\psi^A, G_B\}\eta^B + \frac{1}{2}\{\psi^A, U^D_{BC}\}\eta^B\eta^C\bar{\mathcal{P}}_D \ ,$$

and

$$\{\bar{\mathcal{P}}_A, Q_{\mathcal{B}}\} = G_B\{\bar{\mathcal{P}}_A, \eta^B\} + \frac{1}{2}U^D_{BC}\{\bar{\mathcal{P}}_A, \eta^B\eta^C\}\bar{\mathcal{P}}_D \ , \tag{11.107}$$

or

$$\{\bar{\mathcal{P}}_A, Q_{\mathcal{B}}\} = -G_A + \bar{\mathcal{P}}_D U^D_{AB}\eta^B \ .$$

Here we made use of the antisymmetry of U^C_{AB} in A and B. Hence $\{\Psi, Q_{\mathcal{B}}\}$ is given by

$$\{\Psi, Q_{\mathcal{B}}\} = \bar{\mathcal{P}}_A\{\psi^A, G_B\}\eta^B + \frac{1}{2}\bar{\mathcal{P}}_A\{\psi^A, U^D_{BC}\}\eta^B\eta^C\bar{\mathcal{P}}_D - G_A\psi^A - \bar{\mathcal{P}}_C U^C_{AB}\psi^A\eta^B \ .$$

Inserting this expression in (11.102), we obtain for the unitarizing Hamiltonian H_U

$$H_U = H_{\mathcal{B}} - G_A\psi^A + \bar{\mathcal{P}}_A\{\psi^A, G_B\}\eta^B + \frac{1}{2}\bar{\mathcal{P}}_A\{\psi^A, U^D_{BC}\}\eta^B\eta^C\bar{\mathcal{P}}_D - \bar{\mathcal{P}}_C U^C_{AB}\psi^A\eta^B \tag{11.108}$$

where for a theory of rank one, $H_{\mathcal{B}}$ is given by (11.106). In particular, for q and p (i.e. field) independent structure functions U_{AB}^C this expression reduces to

$$H_U = H_{\mathcal{B}} + \bar{\mathcal{P}}_A\{\psi^A, G_B\}\eta^B - G_A\psi^A - \bar{\mathcal{P}}_C U_{AB}^C \psi^A\eta^B \ ; \ \ (U_{AB}^C \ \ q,p \ independent) \tag{11.109}$$

Note that the information about the theory of interest is embodied in H (contained in $H_{\mathcal{B}}$), as well as in algebraic properties associated with the constrained structure, while the freedom in the choice of gauge is contained entirely in the BRST exact term, i.e., in the last four terms of (11.108).

11.5.4 Connecting Z_Ψ with the quantum partition function in a physical gauge. The $SU(N)$ Yang-Mills theory

Let us now come to the important question, whether a given partition function, having the structure dictated by the FV principal theorem, is the correct quantum partition function corresponding to a given classical Lagrangian. As we have already mentioned this must be checked in each case separately.

As an example we consider the $SU(N)$ Yang-Mills theory described by the Lagrangian (11.59) and canonical Hamiltonian (11.64). Choose ψ^A in (11.108) to be of the form

$$\psi^A := (\vec{\chi}(\vec{A}), \vec{0}) \, , \tag{11.110}$$

where $\vec{\chi}$ and $\vec{0}$ are (N^2-1)-component vectors in "colour space", depending only on the spacial components of the gauge fields. Below we shall simply denote them by $\chi^a(x)$. Since the structure functions $\tilde{U}_{AB}^C$, defined in (11.70), are field independent in this case, we have from (11.109), (11.110) and (11.68), (11.70) and (11.72), [23]

$$\begin{aligned} H_U = H_0 &+ \int d^3x \ \eta^A \tilde{V}_A^B \bar{\mathcal{P}}_B - \int d^3x \ \chi^a \pi_0^a \\ &+ \int d^3x d^3y \ \bar{c}_a(x)\{\chi^a(x), T_b(y)\}c^b(y) \end{aligned}$$

with the constraints (11.68). Taking H_0 to be the canonical Hamiltonian (11.64), the corresponding structure functions $\tilde{V}_A^B$ are given by the matrix (11.72). Then H_U takes the form

$$\begin{aligned} H_U = H_0 &- \int d^3x \ \left(\chi^a \pi_0^a - \bar{P}_a P^a + g f_{abc} \bar{P}_a c^b A_0^c\right) \\ &+ \int d^3x d^3y \ \bar{c}_a(x)\{\chi^a(x), T_b(y)\}c^b(y) \ . \end{aligned} \tag{11.111}$$

[23] Recall footnote 19.

Thus H_U is a function of A^a_μ, π^a_μ, the ghost variables c^a, $\bar{c}_a$ and their canonically conjugate momenta, $\bar{P}_a$ and P^a. The unitarizing action is therefore given by

$$S_U = \int d^4x \, \left(\dot{A}^\mu_a \pi^a_\mu + \dot{\bar{c}}_a P^a + \dot{c}^a \bar{P}_a - \mathcal{H}_U \right) \, , \tag{11.112}$$

with the kinetic term ordered in the form "velocity $\times$ momentum". This leaves us with the problem of choosing $\chi^a(\vec{A})$. But let us first briefly demonstrate how the FV principal theorem leads, for an appropriately chosen fermion gauge fixing function Ψ, to the Faddeev-Popov result for the Yang-Mills theory in the Lorentz gauge [Henneaux 1985].

Given a fermion gauge fixing function of the form (11.110) one can perform the integrations over P and $\bar{P}$ in the BFV partition function. This partition function is given by,

$$Z = \int DAD\pi DcD\bar{c} \; e^{i\int d^4x[\dot{A}^\mu_a\pi^a_\mu - \mathcal{H}_0 + \chi^a\pi^a_0 - \int d^4y \;\bar{c}_a(x)\{\chi^a(x), T_b(y)\}c^b(y)]} \hat{Z}[c,\bar{c}] \, , \tag{11.113}$$

where

$$\hat{Z} = \int D\bar{P}DP e^{i\int d^4x\left[\dot{\bar{c}}_a P^a + \bar{P}_a(-\dot{c}^a - P^a + gf_{abc}c^bA^c_0)\right]} \, .$$

Choosing

$$\chi^a = \partial^i A^a_i \, , \tag{11.114}$$

and performing the integration over $\bar{P}^a$, (11.113) becomes

$$Z = \int DAD\pi \mathcal{D}\bar{c}\mathcal{D}c \; e^{i\int d^4x\left[\dot{A}^\mu_a\pi^a_\mu - \mathcal{H}_0 + (\partial^i A^a_i)\pi^a_0 - (\partial_i\bar{c}_a)\mathcal{D}^i_{ab}c^b\right]} \hat{Z}[c,\bar{c}] \, , \tag{11.115}$$

where

$$\hat{Z} = \int DP \, \prod_{\vec{x},a} \delta\left(P^a + \mathcal{D}^0_{ab}c^b\right) e^{i\int d^4x \dot{\bar{c}}_a P^a} \, . \tag{11.116}$$

Here $\mathcal{D}^0_{ab}$ is the zeroth component of the covariant derivative (11.78), and we have made a partial integration along the way. Hence after integrating over P^a we arrive at

$$\hat{Z} = e^{-i\int d^4x \; \dot{\bar{c}}^a \mathcal{D}^0_{ab} c^b} \, .$$

This factor combines in (11.115) with the exponential and leads to the replacement $(\partial_i\bar{c}_a)\mathcal{D}^i_{ab}c^b$ by $(\partial_\mu\bar{c}_a)\mathcal{D}^\mu_{ab}c^b$. Finally, carrying out also the π, c and $\bar{c}$ integrations, one recovers the partition function in the Lorentz gauge as obtained using the Faddeev-Popov trick, i.e., eq. (11.80).

Our result does not yet imply that the Faddeev-Popov result correctly describes the $SU(N)$ Yang-Mills theory. For this we must still demonstrate that for gauge invariant correlation functions, the above partition function is equivalent to the well known partition function in the Coulomb gauge. This we now show.

Consider once more the partition function in the form (11.103). Since according to the FV theorem this partition function does not depend on the choice of the fermion gauge fixing function Ψ, we can replace χ^a by [Faddeev 1977]

$$\chi'^a = \frac{1}{\varepsilon}\partial^i A_i^a(x) \ , \tag{11.117}$$

where ε is an arbitrary parameter. By the FV theorem the partition function will not depend on ε.

The next step consists in making a change of variables. Thus introducing the variables, $\pi_0'^a \equiv \frac{1}{\varepsilon}\pi_0^a$, and $c'^a = \frac{1}{\varepsilon}c^a$, the partition function (11.113) remains unchanged, except for the kinetic contribution $\dot{A}_a^0\pi_0^a$, which becomes multiplied by ε, and an irrelevant ε-dependent factor multiplying the partition function. Hence with the choice (11.117) in (11.113) one is led to the following equivalent form for the partition function,

$$Z = \int D\mu \ e^{i\int d^4x\left[\varepsilon \dot{A}_a^0\pi_0'^a + \dot{A}_a^i\pi_i^a - \mathcal{H}_0' + A_0^a\mathcal{D}_{ab}^i\pi_i^b + \partial_i A_a^i\pi_0'^a - \partial_i\bar{c}_a\mathcal{D}_{ab}^i c'^b\right]}\hat{Z}'[c',\bar{c}]$$

where $D\mu = DAD\pi_0' D\vec{\pi}D\bar{c}Dc'$, and

$$\hat{Z}' = \int DP \ \prod_{\vec{x},a}\delta\left(P^a + \varepsilon\mathcal{D}_{ab}^0 c'^b\right) e^{-i\int d^4x P^a\partial^0\bar{c}_a} \ .$$

$\mathcal{H}'$ is the Hamiltonian density evaluated on the full constrained surface Γ,

$$\mathcal{H}_0' = \frac{1}{2}\sum_{i,a}(\pi_i^a)^2 + \frac{1}{4}\sum_{i,a}(F_{ij}^a)^2 \ .$$

Since the c^a's are Grassmann valued, the Jacobian of the above transformation equals unity. Taking the limit $\varepsilon \to 0$, and integrating over $A_a^0, \pi_0^{'a}$, $c^{'a}$, $\bar{c}_a$, we are then left with the partition function

$$Z_{Coul} = \int D\vec{A}D\vec{\pi}\prod_{a,x}\delta(\mathcal{D}_{ab}^i\pi_i^b(x))\prod_{a,x}\delta(\partial^i A_i^a(x))\det(\partial_i\mathcal{D}_{ab}^i)e^{i\int d^4x \ (\dot{A}_a^i\pi_i^a - \mathcal{H}_0')} \tag{11.118}$$

or

$$Z_{Coul} = \int D\vec{A}D\vec{\pi}\prod_{a,x}\delta(T_a(x))\prod_{a,x}\delta(\chi^a(x))\det\{\chi^a(x), T_b(y)\}e^{i\int d^4x \ (\dot{A}_a^i\pi_i^a - \mathcal{H}_0')}$$

where χ^a is given by (11.114).

Expression (11.118) is the well known partition function in the Coulomb gauge. Hence, by making use of the FV-theorem and some scaling arguments, we have, at least formally, shown that the Faddeev-Popov prescription for obtaining the QCD path integral in the Lorentz gauge is in fact correct.

11.6 Equivalence of the SD and MCS models

In this section we consider once more the self dual (SD) model [Townsend 1984], defined by the Lagrangian (7.34). Since the Lagrangian describes a purely second class system, the starting point for a BRST formulation must be its classical first class embedded version discussed in chapter 7. Having constructed the FV-partition function, we then make use of the freedom in the choice of gauge to prove the equivalence of the SD-model with the Maxwell-Chern-Simons (MCS) theory within its gauge invariant sector [Banerjee 1997b]. This, as we shall see, is not completely straightforward, but involves some subtle steps.

In the case of a first class system, obtained via BFT embedding from a theory with second class constraints, we are just given a Hamiltonian $\tilde{H}$ and a set of constraints which are all in strong involution. Since we have a priori no Lagrangian associated with the embedded model, we can therefore not distinguish between primary and secondary constraints. Hence we cannot simply make use of previously obtained results. But the general principles on which the construction of the quantum partition function rests, such as the nilpotency of the BRST charge as generator of the BRST symmetry, and the general FV form of the partition function (11.103) are still valid.

The partiton function (11.103) was based on the so-called *minimal* BRST operator obtained in the phase space of the original variables, ghosts and antighosts. The BRST operator is however only unique in the sense, that the cohomology always contains the observables of the theory. Its representation however depends on the choice of the underlying phase-space, which need not be identical with the above mentioned minimal phase-space, as long as it does not change the algebra of the observables. It turns out that in certain cases, quantization of the theory requires an extension of this original phase space (see comment later on).

Our starting point for proving the above stated equivalence between the SD-model and the MCS-theory is the first class embedded version of the self dual (SD) model discussed in subsection 4.3 of chapter 7. There we have seen that the original four second class constraints were converted into a set of four first class constraints by introducing four fields ϕ^α, one for each constraint, satisfying the simplectic algebra (7.1). The new first class constraints, obtained from the second class constraints Ω_α, we denoted by $\tilde{\Omega}_\alpha$ ($\alpha = 0, 1, 2, 3$) and are given by (7.41). The new Hamiltonian $\tilde{H}$, in strong involution with the constraints $\tilde{\Omega}_\alpha$, was shown to have the form (7.42). Since this Hamiltonian and the first class constraints are in strong involution, it follows that the corresponding structure functions defined by equations analogous to (11.105) all vanish. Hence in this case the BRST Hamiltonian $H_{\mathcal{B}}$ equals $\tilde{H}$, so that the corresponding unitarizing Hamiltonian is

$$H_U = \tilde{H} + \int d^2x \, \{\Psi(x), Q_{\mathcal{B}}\} \ , \tag{11.119}$$

where,

$$\tilde{H} = \int d^2x \left(\frac{1}{2} F_0^2 + \frac{1}{2} \vec{F}^2 - m F^0 \tilde{\Omega}_3 \right) \ , \tag{11.120}$$

with $F^0 = f^0 + \phi^3$ and $f^i = f^i + \phi^i$.

Our aim is now to construct a gauge invariant partition function having the structure dictated by the FV-theorem (11.103), and having the property that in two appropriately chosen gauges it reduces to either the partition function of the SD-model, or to that of the MCS theory. For this it turns out to be necessary to double the gauge degrees of freedom. The corresponding additional generators we denote by $\tilde{\pi}_\alpha$ ($\alpha = 1, \ldots, 4$), and are assumed to be in strong involution with the remaining variables. Being generators of an additional symmetry, we group them together with the $\tilde{\Omega}_\alpha$'s into the vector

$$G_A := (\{\tilde{\pi}_\alpha\}, \{\tilde{\Omega}_\alpha\}) \ . \tag{11.121}$$

Having introduced the $\tilde{\pi}_\alpha$'s, we must also introduce the corresponding conjugate variables λ^α, in order to conform with the FV-theorem. Proceeding as in the case of the Maxwell theory we introduce the vectors

$$\eta^A := \begin{pmatrix} P^\alpha \\ c^\alpha \end{pmatrix} \ , \quad \bar{\mathcal{P}}_A := \begin{pmatrix} \bar{c}_\alpha \\ \bar{P}_\alpha \end{pmatrix} \ , \tag{11.122}$$

with Poisson brackets given by (11.93). The freedom in choosing a gauge in the FV functional integral is then realized via a BRST exact term involving the "non-minimal" BRST charge $Q_{\mathcal{B}}$ of the simple form

$$Q_{\mathcal{B}} = \int d^2x \; \eta^A G_A = \int d^2x \; \left(\tilde{\Omega}_\alpha c^\alpha + \tilde{\pi}_\alpha P^\alpha \right) \ .$$

With the introduction of the new canonical pair, $(\lambda^\alpha, \tilde{\pi}_\alpha)$, we must also include a corresponding term $\dot{\lambda}^\alpha \tilde{\pi}_\alpha$ in the kinetic contribution to the FV action in order to conform with the FV theorem.

Next we must choose a gauge fixing function $\Psi(x)$ in (11.119). This function should carry ghost number $gh(\Psi) = -1$. We therefore make the following ansatz, to be justified a posteriori,

$$\Psi(x) = \bar{\mathcal{P}}_A \psi^A \,, \quad \psi^A := (\chi^\alpha, \lambda^\alpha)\,,$$

where χ^α is a function of the fields $f^\mu(x)$ and $\phi^\beta(x)$ to be fixed below. Hence

$$\begin{aligned}\int d^2x\ \{\Psi(x), Q_{\mathcal{B}}\} = & \int d^2x d^2y\ \bar{c}_\alpha(x)\{\chi^\alpha(x), \tilde{\Omega}_\beta(y)\}c^\beta(y) \\ & - \int d^2x[\lambda^\alpha\tilde{\Omega}_\alpha + \tilde{\pi}_\alpha\chi^\alpha - \bar{P}_\alpha P^\alpha]\,.\end{aligned}$$

The $\bar{P}P$ term, arising from our above extension of phase space, is important in order to make the FV partition function below well defined, and was in fact one motivation for extending our space. Since our algebra of constraints is strongly involutive, it cannot arise from the BRST Hamiltonian $H_{\mathcal{B}} = \tilde{H}$.

Choosing for our canonically conjugate auxiliary fields in the kinetic term to be ϕ^α and $\phi^\beta\omega_{\beta\alpha}$, $(\alpha = 1, 2)$, the FV-partition function takes the form

$$Z_{FV} = \int DfD\pi D\phi D\lambda D\tilde{\pi}D\bar{c}DcD\bar{P}DP\ e^{iS_U}\ , \tag{11.123}$$

where [24]

$$\begin{aligned}S_U = \int\ \Big[&\dot{f}^\mu\pi_\mu + \frac{1}{2}\phi^\alpha\omega_{\alpha\beta}\dot{\phi}^\beta + \dot{\lambda}^\alpha\tilde{\pi}_\alpha + \dot{c}^\alpha\bar{P}_\alpha + \dot{\bar{c}}_\alpha P^\alpha - \mathcal{H}_0(f^0 + \phi^3, f^i + \phi^i) \\ &- \bar{c}_\alpha\{\chi^\alpha, \tilde{\Omega}_\beta\}c^\beta + \lambda^\alpha\tilde{\Omega}_\alpha + \tilde{\pi}_\alpha\chi^\alpha - \bar{P}_\alpha P^\alpha)\Big]\end{aligned} \tag{11.124}$$

and $\mathcal{H}_0(f^0, f^i)$ is defined in (7.38). Note the factor 1/2 in the second term on the rhs, since we are summing over all α and β. As we will see below, the term $\tilde{\pi}_\alpha\chi^\alpha$ will give rise to a gauge condition $\chi^\alpha = 0$ after an appropriate scaling of the integration variables. This was another reason for working in a non-minimal space.

Doing the P and $\bar{P}$ integration we are left with

$$\begin{aligned}Z_{FV} = & \int DfD\pi D\phi D\lambda D\tilde{\pi} e^{i\int\ (\dot{f}^\mu\pi_\mu + \frac{1}{2}\phi^\alpha\omega_{\alpha\beta}\dot{\phi}^\beta + \dot{\lambda}^\alpha\tilde{\pi}_\alpha)} \\ \times & \int D\bar{c}Dc\, e^{i\int\left(-\dot{\bar{c}}\dot{c} - \mathcal{H}_0(f^0+\phi^3, f^i+\phi^i) - \bar{c}_\alpha\{\chi^\alpha, \tilde{\Omega}_\beta\}c^\beta + \lambda^\alpha\tilde{\Omega}_\alpha + \tilde{\pi}_\alpha\chi^\alpha\right)}\ .\end{aligned} \tag{11.125}$$

[24] In order to keep the expressions as simple as possible, we adopt the following convention: if the integrand involves also non-local expressions, then a simple integration sign will also stand for multiple integrations associated with multiple summations over fields in non-local expressions. The dimensionality of the (non-specified) measure will be clear from the (Hamiltonian, or action) content. Thus, for example in (11.124), $\int \bar{c}_\alpha\{\chi^\alpha, \tilde{\Omega}_\beta\}c^\beta$ stands for $\int d^3xd^3y\bar{c}_\alpha(x)\{\chi^\alpha(x), \tilde{\Omega}_\beta(y)\}c^\beta(y)$.

We now proceed as in the Yang-Mills case, and make the replacement $\chi^\alpha \to \frac{1}{\varepsilon}\chi^\alpha$. The corresponding partition function does not depend on ε. Next we make the change of variables $\bar{c}_\alpha = \varepsilon\bar{c}'_\alpha \quad \tilde{\pi}^\alpha = \varepsilon\tilde{\pi}'^\alpha$. The Jacobian for this transformation is unity (recall that the Jacobians differ for Grassmann variables from the usual ones). Taking the limit $\varepsilon \to 0$ in the functional integral, and performing the integrations over all Grassmann variables and "Lagrange mutipliers" $\tilde{\pi}'_\alpha$ and λ^α, one is left with the following gauge fixed partition function

$$Z_{gf}^{FV} = \int DfD\pi D\phi(\prod_{\alpha,x}\delta[\chi^\alpha]\delta[\tilde{\Omega}_\alpha])det\{\chi^\alpha,\tilde{\Omega}_\beta\}$$
$$\times\, e^{i\int\left(\dot{f}^\mu\pi_\mu+\frac{1}{2}\phi^\alpha\omega_{\alpha\beta}\dot{\phi}^\beta-\mathcal{H}_0(f^0+\phi^3,f^i+\phi^i)\right)}\ . \tag{11.126}$$

This partition function is the starting point for proving the claimed equivalence.

Equivalence of Z_{FV} and Z_{SD}

Let us choose $\bar{\chi}^\alpha = \Omega_\alpha$, where Ω_α are the second class constraints (7.36) and (7.37). Looking back at the expressions for the first class constraints (7.41), we see that

$$\prod_\alpha \delta[\Omega_\alpha]\delta[\tilde{\Omega}_\alpha] = \prod_\alpha \delta[\Omega_\alpha]\delta[\phi^\alpha]\ .$$

Hence the ϕ-integration in (11.126) can be trivially performed and we obtain, aside from an irrelevant multiplicative constant,

$$Z_{SD} = \int DfD\pi\ (\prod_{\alpha=0}^{3}\delta[\Omega_\alpha])det\ \{\Omega_\alpha,\Omega_\beta\}e^{i\int d^3x\left(\dot{f}^\mu\pi_\mu-\mathcal{H}_0(f)\right)}\ , \tag{11.127}$$

which is the partition function of the SD-model. Hence the partition function of this model is a particular gauge fixed version of the FV-partition function, thus proving the equivalence of the SD-model with the theory associated with Z_{FV} in (11.123) within the gauge invariant sector.

Equivalence of Z_{FV} and Z_{MCS}

We now make again use of our freedom of gauge choice to prove in turn the equivalence of the theory defined by (11.126), and the MCS theory defined by the Lagrangian (7.54) within the BRST invariant and gauge invariant sectors, respectively. Consider the following alternative choice of gauge conditions in (11.126):

$$
\begin{aligned}
\chi^0 &:= f^1 + \frac{1}{m}\partial^2 f^0 = 0 \ , \\
\chi^i &= \phi^i + \frac{1}{m}\partial^i \phi^0 = 0 \quad (i = 1, 2) \ , \\
\chi^3 &:= f^2 - \frac{1}{m}\partial^1 f^0 = 0 \, .
\end{aligned}
\tag{11.128}
$$

Note that these conditions imply that

$$
\partial_i f^i = 0 \ \ , \ \epsilon_{ij}\partial^i \phi^j = 0 \ . \tag{11.129}
$$

Hence it is a Coulomb like gauge. In the following we shall make extensive use of those properties. Implementing the constraint $\tilde{\Omega}_3 = 0$ in (11.120), one is led to the following expression for the Hamiltonian density

$$
\tilde{\mathcal{H}} = \frac{1}{2}(f^0 + \phi^3)^2 + \frac{1}{2}(f^i + \phi^i)^2
$$

which because of (11.128) can also be written in the form

$$
\tilde{\mathcal{H}} = \frac{1}{2m^2}(\epsilon_{ij}\partial^i f^j)^2 - \frac{1}{2} f^i f_i - \frac{1}{2m^2}\phi^0 \nabla^2 \phi^0 \ ,
$$

where we have dropped a surface term. One then finds, after performing the integrations over the momenta and the ϕ^i's in (11.126), and dropping any surface contributions, that

$$
\begin{aligned}
Z_{gf}^{FV} = &\int Df D\phi^0 \prod_{i,x} \delta[f^i + \frac{1}{m}\epsilon_{ij}\partial^j f^0] \\
&\times e^{i\int d^3x \ \left[\frac{1}{m^2}\phi^0 \epsilon_{ij}\partial^i \partial^0 f^j + \frac{1}{2m^2}\phi^0 \nabla^2 \phi^0 + \frac{1}{2m}\epsilon_{ij} f^i \partial^0 f^j - \frac{1}{2m^2}(\epsilon_{ij}\partial^i f^j)^2 + f^i f_i\right]} \ ,
\end{aligned}
$$

where we have dropped the constant determinant $\det\{\bar{\chi}^\alpha, \tilde{\Omega}_\beta\}$. The above expression can also be rewritten as

$$
\begin{aligned}
Z_{gf}^{FV} = &\int D\mu(f,\phi) \ e^{i\int \ \left(\phi^0 + \epsilon_{ij}\partial^i\partial^0 f^j \frac{1}{\nabla^2}\right)\frac{\nabla^2}{2m^2}\left(\phi^0 + \frac{1}{\nabla^2}\epsilon_{kl}\partial^k\partial^0 f^l\right)} \\
&\times e^{i\int \ \left[-\frac{1}{2m^2}\left(\epsilon_{ij}\partial^i\partial^0 f^j\right)\frac{1}{\nabla^2}\left(\epsilon_{kl}\partial^k\partial^0 f^l\right) - \frac{1}{2m^2}(\epsilon_{ij}\partial^i f^j)^2 + \frac{1}{2m}\epsilon_{ij} f^i \partial^0 f^j + \frac{1}{2m^2} f^0 \nabla^2 f^0\right]} \ ,
\end{aligned}
\tag{11.130}
$$

where

$$
D\mu(f,\phi) = Df D\phi_0 \ \prod_{i,x} \delta[f^i + \frac{1}{m}\epsilon_{ij}\partial^i f^0] \ .
$$

Making use of the relation

$$\epsilon_{ij}\epsilon_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \ ,$$

valid in two dimensions, one finds - upon making use of (11.129) that

$$\begin{aligned}\frac{1}{2m^2}\left(\epsilon_{ij}\partial^i\partial^0 f^j\right)\frac{1}{\nabla^2}\left(\epsilon_{kl}\partial^k\partial^0 f^l\right) &= \partial^0 f^i\partial^0 f_i \\ &= \frac{1}{2m^2}F^{0i}F_{0i} + \frac{3}{2m^2}f^0\nabla^2 f^0 + s.t.\ ,\end{aligned}$$

where "s.t." stands for "surface term", and

$$F^{\mu\nu} \equiv \partial^\mu f^\nu - \partial^\nu f^\mu \ .$$

Furthermore we have that

$$\frac{1}{2m}\epsilon_{ij}f^i\partial^0 f^j = -\frac{1}{4m}\epsilon_{i0j}f^iF^{0j} - \frac{1}{4m}\epsilon_{ij0}f^iF^{j0} + \frac{1}{4m}\epsilon_{0ij}f^0F^{ij} + s.t.\ ,$$

and because of the gauge conditions on f^i in (11.128)

$$\frac{1}{m^2}f^0\nabla^2 f^0 = -mf^i(\epsilon_{ij}\partial^j f^0) = -\frac{1}{2m}\epsilon_{0ij}f^0F^{ij} + s.t.$$

Here $\epsilon_{\mu\nu\lambda}$, $(\mu,\nu,\lambda = 0,1,2)$ is the ϵ-tensor in three dimensions, with $\epsilon_{012} = 1$. Performing the integration over ϕ^0 and combining the various contributions in (11.130) one finds that

$$Z_{gf'}^{FV} = \int DA\ \prod_{i,x}\delta(mA^i + \epsilon_{ij}\partial^j A^0)\ exp\left(i\int d^3x\ \mathcal{L}_{MCS}\right)\ , \qquad (11.131)$$

where $\mathcal{L}_{MCS}$ is the Lagrangian density of the Maxwell-Chern-Simons theory

$$\mathcal{L}_{MCS} = -\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{m}{2}\epsilon_{\mu\nu\lambda}A^\mu\mathcal{F}^{\nu\lambda}\ ,$$

with $A^\mu \equiv \frac{1}{m}f^\mu$ and $\mathcal{F}^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. This, we claim, is the partition function of the MCS theory in the gauge $A^i + \frac{1}{m}\epsilon_{ij}\partial^i A^0{=}0$.

The MCS theory is a theory in 3 space-time dimensions with the following first class constraints:

$$\pi_0 = 0\,, \quad \partial^i\pi_i + \frac{m}{2}\epsilon_{ij}\partial^i A^j = 0\ .$$

We denote this set by $\Omega_\mu = 0$ $(\mu = 0,1,2)$. The Hamiltonian density is given by

$$\mathcal{H}_{MCS} = \frac{1}{2}\vec{\pi}^2 + \frac{1}{4}F^{ij}F_{ij} - A^0\partial^i\pi_i - \frac{m}{2}\epsilon_{ij}\pi_iA^j - \frac{m^2}{8}A_iA^i - \frac{m}{2}\epsilon_{ij}A^0\partial^iA^j\ ,$$

where summation over repeated indices is understood. Introducing a set of gauge conditions $\chi^\mu = 0, (\mu = 0, 1, 2)$ the partition function for this now second class system is given by

$$Z_{MCS} = \int \mathcal{D}A\mathcal{D}\pi \prod_x \delta[\pi_0] \prod_x \delta\left(\partial^i \pi_i + \frac{m}{2}\epsilon_{ij}\partial^i A^j\right)$$
$$\times \prod_{\mu,x} \delta[\chi^\mu(A)] \det\{\chi^\mu, \Omega_\nu\} e^{i\int d^3x \ (\dot{A}^\mu \pi_\mu - \mathcal{H}_{MCS})}$$

Let us assume that the χ^μ's do not depend on the canonical momenta, and that they depend linearly on the "potentials" A^μ. Then the determinant is just a constant. We can then perform the momentum integrations and obtain

$$Z_{MCS} = \int DA \prod_{\mu,x} \delta[\chi^\mu(A)] \ e^{i\int d^3x \ \mathcal{L}_{MCS}} .$$

Comparing this expression with (11.131) we conclude that the MCS theory in the gauge $A^i + \frac{1}{m}\epsilon_{ij}\partial^j A^0 = 0$, and the second class self dual model described by the partition function (11.127), are just two gauge fixed versions of the BRST invariant FV-partition function (11.126), with the action given by (11.124). [25]

11.7 The physical Hilbert space. Some remarks

So far our discussion has been carried out entirely within the functional approach. We conclude this chapter with some remarks about the Hilbert space underlying the partition function (11.103).

The Faddeev-Popov *ghost charge* Q_{gh} is associated with the rigid symmetry transformation [Kugo 1979]

$$c^a \to e^\epsilon c^a \quad ; \quad \bar{c}_a \to e^{-\epsilon}\bar{c}_a .$$

Its Poisson brackets of c^a and $\bar{c}_a$ with the ghost charge are given by

$$\{c^a, Q_{gh}\} = c^a , \quad \{\bar{c}_a, Q_{gh}\} = -\bar{c}_a$$

where

$$Q_{gh} = c^a \bar{P}_a - \bar{c}_a P^a .$$

Furthermore

$$\{H_{\mathcal{B}}, Q_{gh}\} = 0 , \quad \{Q_{gh}, Q_{\mathcal{B}}\} = 1$$

[25] For an alternative proof, starting from a "master Lagrangian" see [Banerjee 1995b].

These relations state that $Q_{\mathcal{B}}$ and $H_{\mathcal{B}}$ carry ghost numbers one and zero respectively. After mapping the generalized Poisson brackets onto commutators and anti-commutators of the corresponding operators, physical states are obtained upon i) eliminating BRST equivalent states, ii) requiring the absence of ghosts in the physical spectrum and iii) eliminating zero norm states. The first condition requires that

$$\hat{Q}_{\mathcal{B}}|phys\rangle = 0 \ , \tag{11.132}$$

where $\hat{Q}_{\mathcal{B}}$ is now an operator in Hilbert space. Hence physical states are BRST invariant. Similarly the absence of ghost states in the physical spectrum is implemented by the requirement

$$\hat{Q}_{gh}|phys\rangle = 0 \ . \tag{11.133}$$

The space defined by (11.132) and (11.133) still contains zero norm states. Let V_0 be the subspace of zero-norm states. Clearly the norm of any state of the form

$$|\Psi^{(0)}\rangle = \hat{Q}_{\mathcal{B}}|\Psi\rangle$$

vanishes because of the nilpotency of $\hat{Q}_{\mathcal{B}}$, i.e., $\hat{Q}_{\mathcal{B}}^2 = 0$. This property is referred to as BRST cohomology. The zero norm states have no effect on physical processes, i.e. $|\Psi\rangle$ and $|\Psi\rangle + \hat{Q}_B|\Phi\rangle$ are not distinguishable when calculating correlators of BRST invariant operators, if $|\Psi >$ fulfills the conditions (11.132) and (11.133). After eliminating also the zero norm states, the physical S-matrix S_{phys} can be defined consistently in this (positive norm) Hilbert space, and the unitarity of S_{phys} can be proven [Kugo 1979], i.e.,

$$S^\dagger_{phys}S_{phys} = S_{phys}S^\dagger_{phys} = 1.$$

Chapter 12

Field-Antifield Quantization

12.1 Introduction

In the preceding chapters we have systematically developed a Hamiltonian formalism leading us to the quantum phase space partition function in a general gauge. The main drawbacks of this formulation are twofold: i) One generally wants to formulate the Feynman rules in configuration space. ii) The Hamiltonian formulation is not manifestly covariant, so that it is not clear which gauge fixing condition in phase space (or which fermion gauge fixing function) implies a covariant choice of gauge in the Lagrangian formulation.

The field-antifield approach of Batalin and Vilkovisky (BV) to the quantization of theories with a local symmetry [Batalin 1981/83a,d/84] solves both of these problems. In this formulation the solution of the so-called master equation is the configuration-space counterpart of the Fradkin-Vilkovisky (FV) phase-space action. The equivalence of the Hamiltonian BFV formalism and the BV field-antifield formalisms has been studied e.g. in [Fisch 1989, Grigoryan 1991, De Jonghe 1993]. The present chapter is intended as an introduction to the central ideas of this formalism, which we shall motivate by starting from the FV representation of the partition function discussed in chapter 11. For an extensive discussion of the field-antifield formalism, the reader is referred to [Henneaux 1990b, Gomis 1995].

In the following we first present the axiomatic approach of Batalin and Vilkovisky, and then present a proof for systems under some restrictive conditions.

12.2 Axiomatic field-antifield formalism

The field-antifield formalism is usually presented without immediate motivation in an axiomatic way, the justification of the procedure relying ultimately on the observation that the field-antifield action, obtained by solving the so-called master equation in field-antifield space, embodies the full gauge structure of the classical action in question. [1] In this approach to quantization one departs from the classical configuration-space action $S_{cl}[q]$, which is a functional of the coordinates q^i, $i = 1, \cdots, n$, and is invariant under a set of gauge transformations,

$$\delta q^i(t) = \int dt'\, R^i_\alpha(t,t')\varepsilon^\alpha(t'), \quad \alpha = 1, \cdots, m_0\,, \tag{12.1}$$

where the R^i_α's are in general local functions of the (time-dependent) coordinates, and of explicit time. Suppose that one has found all local symmetry transformations, and that the R^i_α's are linearly independent (i.e., possess no right zero modes), then the theory is said to be *irreducible*, and we are led to m_0 independent Noether identities, [2]

$$\int dt' \frac{\delta S_{cl}}{\delta q^i(t')} R^i_\alpha(t',t) \equiv 0 \quad , \quad \alpha = 1, \cdots, m_0\,. \tag{12.2}$$

If on the other hand the matrix ${R^i}_\alpha$ possesses m_0 right zero modes, then the theory is said to be reducible, and the number of arbitrary gauge parameters is less than m_0. [3]

In the following we restrict ourselves to the irreducible case. For every gauge parameter ε^α we then introduce a ghost field c^α. The complete set of fields we denote by $\vartheta^\ell := (q^i, c^\alpha)$. Note that no anti-ghosts have been introduced at this stage. Following Batalin and Vilkovisky [Batalin 1981] one now introduces for every field ϑ^ℓ an antifield $\vartheta^\star_\ell$, with opposite statistics to that of ϑ^ℓ,

$$\epsilon(\vartheta^\ell) = \epsilon_\ell\,, \quad \epsilon(\vartheta^\star_\ell) = \epsilon_\ell + 1 \quad (mod\ 2)\ ,$$

and ghost number

$$gh(\vartheta^\star_\ell) = -gh(\vartheta^\ell) - 1\ .$$

[1] See the reviews by Henneaux and Gomis [Henneaux 1990b, Gomis 1995].

[2] Note that we can always add to the rhs of (12.1) terms of the form

$$\int dt' \frac{\delta S_{cl}}{\delta q^j(t')} T^{ji}(t',t)$$

with $T^{ij} = -T^{ji}$ (bosonic case), without altering the Noether identity (12.2). Such terms do not contribute to δS_{cl} and are called "trivial" gauge transformations. They do not lead to conserved currents.

[3] For a discussion of this case confer e.g. [Gomis 1995].

In this space of fields and antifields one defines an odd symplectic structure called *antibracket* [Zin-Justin 1975, Batalin 1981]: [4]

$$(f, g) = \int dt \left(\frac{\delta^{(r)} f}{\delta \vartheta^\ell(t)} \frac{\delta^{(l)} g}{\delta \vartheta^\star_\ell(t)} - \frac{\delta^{(r)} f}{\delta \vartheta^\star_\ell(t)} \frac{\delta^{(l)} g}{\delta \vartheta^\ell(t)} \right) , \tag{12.3}$$

where f and g are in general functionals of the fields and antifields, and $\frac{\delta^{(r)}}{\delta\vartheta}$, $\frac{\delta^{(l)}}{\delta\vartheta}$ denote the right- and left-functional derivatives. The summation over ℓ includes an integration over spacial coordinates in the case of a field theory. Recall that for bosonic (B) and fermionic (F) functionals of bosonic (fermionic) variables b (f) we have,

$$\begin{aligned} \frac{\delta^{(l)} F}{\delta f} &= \frac{\delta^{(r)} F}{\delta f} , \quad \frac{\delta^{(l)} F}{\delta b} = \frac{\delta^{(r)} F}{\delta b} \\ \frac{\delta^{(l)} B}{\delta f} &= -\frac{\delta^{(r)} B}{\delta f} , \quad \frac{\delta^{(l)} B}{\delta b} = \frac{\delta^{(r)} B}{\delta b} . \end{aligned} \tag{12.4}$$

The antibracket has the following properties:

i) Grassmann signature

$$\epsilon_{(f,g)} = \epsilon_f + \epsilon_g + 1 \quad (mod\ 2)$$

ii) Commutative and assocative properties:

$$\begin{aligned} (g, f) &= -(-1)^{(\epsilon_f+1)(\epsilon_g+1)} (f, g) , \\ (fg, h) &= f(g, h) + (-1)^{\epsilon_g(\epsilon_h+1)} (f, h) g , \\ (f, gh) &= (f, g) h + (-1)^{\epsilon_g(\epsilon_f+1)} g (f, h) . \end{aligned} \tag{12.5}$$

iii) Jacobi identity

$$\begin{aligned} (-1)^{(\epsilon_f+1)(\epsilon_h+1)} ((f, g), h) &+ (-1)^{(\epsilon_g+1)(\epsilon_f+1)} ((g, h), f) \\ &+ (-1)^{(\epsilon_h+1)(\epsilon_g+1)} ((h, f), g) = 0 . \end{aligned} \tag{12.6}$$

An important property of the antibracket is, that in analogy to the Poisson bracket in phase space, the infinitesimal transformation $\vartheta^\ell \to \vartheta^\ell + \varepsilon(\vartheta^\ell, F)$, $\vartheta^\star_\ell \to \vartheta^\star_\ell + \varepsilon(\vartheta^\star_\ell, F)$, with ε an infinitesimal bosonic parameter, F an arbitrary function of ϑ^ℓ and $\vartheta^\star_\ell$, and $\epsilon(F) = 1$, $gh(F) = -1$, maintains the canonical

[4] Functional derivatives will be understood to be partial derivatives when acting on ordinary functions.

structure $(\vartheta^\ell, \vartheta^\star_k) = \delta^\ell_k$ up to order $O(\epsilon^2)$. [5] From the definition (12.3) one readily verifies that for bosonic functionals B

$$(B, B) = 2 \int dt \, \frac{\delta^{(r)} B}{\delta \vartheta^\ell(t)} \frac{\delta^{(l)} B}{\delta \vartheta^\star_\ell(t)} \, ,$$

and for fermionic functionals F,

$$(F, F) = 0 \, .$$

Furthermore

$$((f, f), f) = 0$$

for any f, bosonic or fermionic.

Consider now a bosonic functional $S[\vartheta, \vartheta^\star]$ with the dimensions of an action, obeying the so-called *classical master equation*:

$$(S, S) = 0 \, , \tag{12.7}$$

with the boundary condition

$$S[\vartheta, \vartheta^\star]_{\vartheta^\star = 0} = S_{cl}[q] \, .$$

Consider further the following transformation of ϑ^ℓ and $\vartheta^\star_\ell$ induced by S:

$$\overset{\vee}{s} \vartheta^\ell \equiv (\vartheta^\ell, S) \, , \quad \overset{\vee}{s} \vartheta^\star_\ell \equiv (\vartheta^\star_\ell, S) \, , \quad \ell = 1, \cdots, N \, . \tag{12.8}$$

Under these transformations S is invariant, since it is a solution to the master equation (12.7). Furthermore, S can be regarded as a nilpotent operator in the antibracket sense, just as the BRST charge $Q_{\mathcal{B}}$ is nilpotent in the Poisson bracket sense: $\{Q_{\mathcal{B}}, Q_{\mathcal{B}}\} = 0$.

Let us collect the fields and antifields into $z^A = (\vartheta^\ell, \vartheta^\star_\ell)$. By defining the matrix

$$\xi^{AB} := \begin{pmatrix} 0 & \delta^\ell_k \\ -\delta^k_\ell & 0 \end{pmatrix}$$

with the symmetry property

$$\xi^{AB} = -\xi^{BA} \, ,$$

[5] In the case where F is an F_2 type generating function, depending on ϑ and the *transformed* antifield $\tilde{\vartheta}^\star_\ell$ with

$$\tilde{\vartheta}^\ell = \frac{\partial F_2}{\partial \tilde{\vartheta}^\star_\ell} \, , \quad \vartheta^\star_\ell = \frac{\partial F_2}{\partial \vartheta^\ell} \, ,$$

the canonical structure of the antibrackets turns out to be preserved exactly [Troost 1990], i.e., $(\tilde{\vartheta}^\ell, \tilde{\vartheta}^\star_{\ell'}) = \delta_{\ell\ell'}$, $(\tilde{\vartheta}^\ell, \tilde{\vartheta}^{\ell'}) = (\tilde{\vartheta}^\star_\ell, \tilde{\vartheta}^\star_{\ell'}) = 0$.

we can write the antibracket (f,g) in the form of an odd symplectic structure:

$$(f,g) = \int dt \frac{\delta^{(r)} f}{\delta z^A(t)} \xi^{AB} \frac{\delta^{(l)} g}{\delta z^B(t)} \, .$$

Since ξ^{AB} only connects antifields with fields, we also have the following relation

$$\xi^{AB} \neq 0 \ \longrightarrow (-1)^{(\epsilon_A+1)(\epsilon_B+1)} = 1 \ . \tag{12.9}$$

Assuming that a solution to (12.7) has been found, we can right-differentiate (12.7) with respect to $z^C(t)$:

$$\int dt' \Big[\frac{\delta^{(r)} S}{\delta z^A(t')} \ \xi^{\ AB} \left(\frac{\delta^{(r)}}{\delta z^C(t)} \frac{\delta^{(l)} S}{\delta z^B(t')} \right) + (-1)^{\epsilon_C \epsilon_B} \left(\frac{\delta^{(r)}}{\delta z^C(t)} \frac{\delta^{(r)} S}{\delta z^A(t')} \right) \xi^{AB} \frac{\delta^{(l)} S}{\delta z^B(t')} \Big] = 0 \, .$$

Making use of

$$\frac{\delta^{(r)}}{\delta z^C(t)} \frac{\delta^{(r)} S}{\delta z^A(t')} = (-1)^{\epsilon_A} \frac{\delta^{(l)}}{\delta z^A(t')} \frac{\delta^{(r)} S}{\delta z^C(t)}$$

and

$$\frac{\delta^{(r)}}{\delta z^C(t)} \frac{\delta^{(l)} S}{\delta z^A(t')} = \frac{\delta^{(l)}}{\delta z^A(t')} \frac{\delta^{(r)} S}{\delta z^C(t)}$$

we obtain, using (12.9),

$$\frac{\delta^{(r)}}{\delta z^B(t)} (S,S) = 2 \int dt' \frac{\delta^{(r)} S}{\delta z^A(t')} \mathcal{R}^A_B(t',t) \equiv 0 \, , \quad A,B = 1, \cdots, 2N \, ,$$

with

$$\mathcal{R}^A_B(t',t) = \xi^{AC} \frac{\delta^{(l)}}{\delta z^C(t')} \frac{\delta^{(r)} S}{\delta z^B(t)} \, . \tag{12.10}$$

These identities are the analog of the gauge identities (12.2), stating the invariance of the field-antifield action under the transformations

$$\delta z^A(t) = \int dt' \mathcal{R}^A_B(t,t') \epsilon^B(t') \ .$$

Thus it appears that S possesses $2N$ gauge degrees of freedom. Not all of them are however independent. Of particular interest are the so-called proper solutions to the master equation, where the number of true gauge degrees of freedom is just N, with N the number of star-variables. This, in fact, is the maximum number of possible gauge degrees of freedom [Gomis 1995]. In this case N gauge degrees of freedom can be eliminated by imposing N gauge conditions. This allows one to eliminate the N antifields in favour of the ϑ^ℓ's, leading to a well defined partition function in the original space of fields.

The so-called *proper* solutions to the master equation contain all the information about the gauge algebra associated with S_{cl}. [6] A particularly simple solution is obtained in the case of *irreducible* theories of rank *one* with a closed gauge algebra and constant structure functions. As discussed in subsection 5 of chapter 2, a transformation of the coordinates of the form (12.1) is a symmetry of the action, if the coefficient functions $R^i_\alpha(t)$ satisfy (12.2). Or, absorbing the time dependences - as was also done in subsection 5 of chapter 2 - into the discrete indices $((i,t) \to i;\ (i,\alpha) \to \alpha$, etc.), with summations implying a corresponding integration over time, we have

$$\frac{\delta S_{cl}}{\delta q^i} R^i_\alpha = 0 \ . \tag{12.11}$$

This is a particularily useful notation if the matrix R^i_α is non-local in time. Furthermore, up to trivial gauge transformations - the closed gauge algebra (2.42) translates into the following algebra for the matrix elements ${R^i}_\alpha$, [7]

$$\frac{\delta R^i_\alpha}{\delta q^j} R^j_\beta - \frac{\delta R^i_\beta}{\delta q^j} R^j_\alpha = R^i_\gamma f^\gamma_{\alpha\beta} \ . \tag{12.12}$$

Let us now construct the solution to the master equation. Assume that S_{cl} is invariant under the transformations $\overset{\vee}{s} q^i = R^i_\alpha c^\alpha$, with R^i_α satisfying (12.11), and the c^α's Grassmann valued. According to (12.8) $\overset{\vee}{s} q^i$ should also be given by (q^i, S). The field-antifield action is therefore of the form

$$S = S_{cl} + q^\star_i R^i_\alpha c^\alpha + S_2[q, c, c^\star] \ . \tag{12.13}$$

With this ansatz

$$\overset{\vee}{s} S_{cl} = (S_{cl}, S) = \frac{\delta S_{cl}}{\delta q^i} R^i_\alpha c^\alpha = 0,$$

because of (12.11). Let

$$S_1 \equiv S_{cl} + q^\star_i R^i_\alpha c^\alpha \ .$$

Then $(S,S) = (S_1,S_1)+(S_1,S_2)+(S_2,S_1)+(S_2,S_2)$. A straightforward algebra yields

$$(S_1, S_1) = q^\star_i \left(\frac{\delta R^i_\alpha}{\delta q^j} R^j_\beta - \frac{\delta R^i_\beta}{\delta q^j} R^j_\alpha \right) c^\alpha c^\beta \ ,$$

[6] See the review by Henneaux and Gomis [Henneaux 1990b, Gomis 1995].

[7] We take the generators to be those of a bosonic theory.

or making use of (12.12),

$$(S_1, S_1) = q_i^\star f_{\alpha\beta}^\gamma R_\gamma^i c^\alpha c^\beta \ .$$

In the case of an abelian gauge algebra, $f_{\alpha\beta}^\gamma = 0$, and S would be given by the first two terms in (12.13). In the non-abelian case the contributions to (S, S) arising from the presence of S_2 in (12.13) must cancel with (S_1, S_1), if S is to be a solution to the master equation. Now S_2 must carry vanishing ghost number and vanishing Grassmann parity. Furthermore it must be a function of $c_\alpha^\star$ and the fields c^α, in order to give a non-vanishing contribution to (S, S). Since the ghost number of $c_\alpha^\star$ is -2, the simplest ansatz is of the form

$$S_2 = -\frac{1}{2}\bar{f}_{\alpha\beta}^\gamma c_\gamma^\star c^\alpha c^\beta \ ,$$

where the $\bar{f}_{\alpha\beta}^\gamma$'s are constants. A bit of algebra shows that by choosing $\bar{f}_{\alpha\beta}^\gamma = f_{\alpha\beta}^\gamma$, and making use of

$$f_{\alpha\rho}^\lambda f_{\beta\gamma}^\rho c^\alpha c^\beta c^\gamma = 0 \ ,$$

following from the Jacobi identity for the structure functions, i.e.,

$$f_{\alpha\beta}^\lambda f_{\lambda\gamma}^\delta + f_{\beta\gamma}^\lambda f_{\lambda\alpha}^\delta + f_{\gamma\alpha}^\lambda f_{\lambda\beta}^\delta = 0 \ ,$$

the field-antifield action

$$S[q, c; q^\star, c^\star] = S_{cl}[q] + \left(q_i^\star R_\alpha^i c^\alpha - \frac{1}{2} c_\alpha^\star f_{\beta\gamma}^\alpha c^\beta c^\gamma \right) \tag{12.14}$$

is a solution to the classical master equation (12.7). Note that the first term under the integral only requires the knowledge of the symmetries of the *classical* action, while the second term reflects a non-trivial algebra of the gauge generators.

From (12.1) and the above solution for S one sees that (12.14) can also be written in the form

$$S = S_{cl} + (q_i^\star \check{s} q^i + c_\alpha^\star \check{s} c^\alpha) \tag{12.15}$$

where

$$\check{s} q^i = R_\alpha^i c^\alpha \ , \quad \check{s} c^\alpha = -\frac{1}{2} f_{\beta\gamma}^\alpha c^\gamma \ .$$

We recognize in these transformations the BRST symmetry transformations of the effective action associated with a classical Lagrangian, for which the generators ${R^i}_\alpha$ satisfy the algebra (12.12).

The solution to the master equation is not unique. Indeed, since (12.14) does not contain the antighosts $\bar{c}_\alpha$, we can always trivially add a term of the form $\bar{c}_\alpha^\star B^\alpha$ to (12.14), without violating (12.7):

$$S[q, c; q^\star, c^\star] \to \bar{S}[q, c, \bar{c}; q^\star, c^\star, \bar{c}^\star; B] = S_{cl}[q] + (q_i^\star \check{s} q^i + c_\alpha^\star \check{s} c^\alpha + \bar{c}_\alpha^\star B^\alpha) \ . \tag{12.16}$$

This solution continues to be proper. B^α will turn out to play the role of the gauge fixing Nakanishi-Lautrup field. The last term in (12.16) will in fact be important when constructing the quantum action in a fixed gauge. Note that at this point $\bar{S}$ does not depend on the antighosts familiar from the Faddeev-Popov trick for implementing a gauge condition. Indeed, the dependence on $\bar{c}_\alpha$ enters indirectly via the gauge fixing, which is realized in [Batalin 1981], seemingly ad hoc, by eliminating the antifields in favour of the fields according to [8]

$$q^\star_{\underline{a}} = \frac{\delta \Psi_L}{\delta q^{\underline{a}}}\,, \quad c^\star_\alpha = \frac{\delta \Psi_L}{\delta c^\alpha}\,, \quad \bar{c}^{\star\alpha} = \frac{\delta \Psi_L}{\delta \bar{c}_\alpha}\,. \tag{12.17}$$

Here $\Psi_L[q, c, \bar{c}]$ is some functional of the fields, with Grassmann parity $\epsilon(\Psi_L) = 1$ and ghost number $gh(\Psi_L) = -1$, properly chosen, such that the corresponding gauge fixed action S_{gf} is non-degenerate. The antighosts $\bar{c}_\alpha$ are needed for constructing a gauge fixing fermion with ghost number $gh(\Psi) = -1$. Implementing (12.17) in (12.16) yields the gauge fixed Lagrangian action

$$S_{gf}[q, c, \bar{c}; B] = \bar{S}[q, c, \bar{c}; \frac{\delta \Psi_L}{\delta q}, \frac{\delta \Psi_L}{\delta c}, \frac{\delta \Psi_L}{\delta \bar{c}}; B]\,.$$

The corresponding partition function then reads,

$$Z^L_\Psi = \int DB \int DqDcD\bar{c}\; e^{iS_{gf}[q,c,\bar{c};B]}\,.$$

Example

Consider the classical action,

$$S_{cl} = \int dt\; \left(\frac{1}{2}\dot{x}^2 + \dot{x}y + \frac{1}{2}(x-y)^2\right) \tag{12.18}$$

which is invariant under the infinitesimal transformations

$$\delta x = \epsilon\,, \quad \delta y = \epsilon - \dot{\epsilon}\,. \tag{12.19}$$

In this example the classical symmetry alone determines already the field-antifield action, which is linear in the antifields. With ϵ in (12.19) replaced by c, we have that

$$\overset{\vee}{s}x = c\,, \quad \overset{\vee}{s}y = c - \dot{c}\,.$$

[8] In the literature the notation Ψ is used for the gauge fixing fermion. We use Ψ_L for the *Lagrange* gauge fixing fermion in order to distinguish it from the *Hamiltonian* gauge fixing fermion Ψ_H in the following section.

Hence according to (12.15) the field-antifield action is given by

$$S = S_{cl} + \int dt\ [x^\star c + y^\star (c - \dot{c})] + \int dt\, \bar{c}^\star B\ . \tag{12.20}$$

One readily checks that S is a solution to the classical master equation. From (12.8) we deduce the remaining transformation laws that leave $S[\vartheta, \vartheta^\star]$ invariant, and hence are nilpotent in the antibracket sense:

$$\begin{aligned}
&\overset{\vee}{s} x^\star = \ddot{x} + \dot{y} - x + y\ , \\
&\overset{\vee}{s} y^\star = -\dot{x} + x - y\ , \\
&\overset{\vee}{s} c = 0\,, \quad \overset{\vee}{s} B = 0\ , \\
&\overset{\vee}{s} \bar{c} = B\ , \quad \overset{\vee}{s} \bar{c}^\star = 0\ .
\end{aligned}$$

We eliminate the star variables by introducing an appropriate set of gauge conditions via (12.17). With the gauge fixing fermion $\Psi_L = \int dt\, \bar{c}(\chi + \dot{y})$, the star variables are evaluated to be

$$\begin{aligned}
&x^\star = \bar{c}\frac{\partial \chi}{\partial x}\ , \\
&y^\star = -\dot{\bar{c}} + \bar{c}\frac{\partial \chi}{\partial y}\ , \\
&c^\star = 0\,, \quad \bar{c}^\star = \dot{y} + \chi\ .
\end{aligned}$$

Substituting these expressions into (12.20) yields, after a partial integration, the gauge fixed action

$$S_{gf} = S_{cl} + \int dt \left(\bar{c}(\dot{c} - \ddot{c}) + \bar{c}\frac{\partial \chi}{\partial y}(c - \dot{c}) + \bar{c}\frac{\partial \chi}{\partial x} c + B(\dot{y} + \chi) \right)\ ,$$

in agreement with (11.20) and (11.21). One readily checks that $(S_{gf}, S) = 0$.

12.3 Constructive proof of the field-antifield formalism for a restricted class of theories

The solution (12.14) is linear in the antifields. This is a characteristic of first rank theories with constant structure constants. For these types of theories the equivalence of the above axiomatic field-antifield approach to the systematic FV Hamiltonian approach of chapter 11 has been discussed starting either from the field-antifield action, or from the BFV Hamiltonian formulation [Batlle 1989/90, Dresse 1991, Grigoryan 1991, De Jonghe 1993, Rothe 2008]. Since this book is primarily concerned with Hamiltonian methods, we will take, in the following, the latter path [Rothe 2008].

Starting from the Fradkin-Vilkovisky (FV) phase space action, we first show that this action can be obtained as a solution to a Hamiltonian master equation in phase space, analogous to (12.7). At this stage there will be no restriction whatsoever on the gauge theory considered. This suggests that an analogous statement should hold on Lagrangian level. To actually find the corresponding Lagrangian version of the master equation in the BV formulation, we must however be able to perform the momentum integrations. Since this cannot be done in general, it restricts the class of gauge theories that can be studied generically.

We now begin with the first part of the above mentioned program, which does not involve any restrictions on the type of theories considered. It merely makes use of the BRST invariance of the phase space action under transformations generated by a nilpotent charge.

12.3.1 From the FV-phase-space action to the Hamiltonian master equation

Consider the FV (unitarized) action in (11.103), with H_U given by (11.104), i.e.,

$$S_U[q,p,\eta,\bar{\mathcal{P}}] = \int dt \left(\dot{q}^i p_i + \dot{\eta}^A \bar{\mathcal{P}}_A - H_{\mathcal{B}}(q,p,\eta,\bar{\mathcal{P}}) + \{\Psi, Q_{\mathcal{B}}\}\right) ,$$

with $H_{\mathcal{B}}$ the BRST Hamiltonian, $Q_{\mathcal{B}}$ the nilpotent BRST charge, and Ψ an arbitrary "fermion gauge-fixing function" [9] depending on the phase-space variables q^i, p_i, η^A, $\bar{\mathcal{P}}_A$. In what follows it will be convenient to write the BRST exact contribution in the form:

$$\{\Psi, Q_{\mathcal{B}}\} = \left(\frac{\partial\Psi}{\partial q^i}\{q^i, Q_{\mathcal{B}}\} + \frac{\partial\Psi}{\partial p_i}\{p_i, Q_{\mathcal{B}}\}\right) + \left(\frac{\partial\Psi}{\partial \eta^A}\{\eta^A, Q_{\mathcal{B}}\} + \frac{\partial\Psi}{\partial \bar{\mathcal{P}}_A}\{\bar{\mathcal{P}}_A, Q_{\mathcal{B}}\}\right) \tag{12.21}$$

where $\{\ ,\}$ is the graded Poisson bracket (11.43), and where we made use of the fact that, according to (12.4), there is not distinction between left- and right-derivatives for fermionic functionals.

The gauge fixing fermion Ψ is an a priori arbitrary function of the fields with Grassmann parity $\epsilon(\Psi) = 1$ and ghost number $gh(\Psi) = -1$, reflecting the

[9] In comparison with (11.104) we have changed the sign of the BRST exact term, in order to facilitate comparison with the literature.

arbitrariness in the choice of gauge. Let us consider the partial derivatives of Ψ as new variables:

$$
\begin{aligned}
q_i^\star(t) &= \frac{\partial \Psi}{\partial q^i(t)}\,, \quad p^{\star i}(t) = \frac{\partial \Psi}{\partial p_i(t)}\,, \\
\eta_A^\star(t) &= \frac{\partial \Psi}{\partial \eta^A(t)}\,, \quad \bar{\mathcal{P}}^{\star A}(t) = \frac{\partial \Psi}{\partial \bar{\mathcal{P}}_A(t)}\,.
\end{aligned}
\tag{12.22}
$$

The above derivatives are functions of q^i, p_i, η^A and $\bar{\mathcal{P}}_A$. For later convenience it is useful to consider the fermion gauge fixing function to be a *functional* of these variables. We denote it by Ψ_H (the subscript H stands for "Hamiltonian"):

$$
\Psi_H[q,p,\eta,\bar{\mathcal{P}}] = \int dt\, \Psi(q(t),p(t),\eta(t),\bar{\mathcal{P}}(t))\,. \tag{12.23}
$$

Then the definitions (12.22) are replaced by expressions involving functional derivatives as follows:

$$
\begin{aligned}
q_i^\star(t) &= \frac{\delta \Psi_H}{\delta q^i(t)}\,, \quad p^{\star i}(t) = \frac{\delta \Psi_H}{\delta p_i(t)}\,, \\
\eta_A^\star(t) &= \frac{\delta \Psi_H}{\delta \eta^A(t)}\,, \quad \bar{\mathcal{P}}^{\star A}(t) = \frac{\delta \Psi_H}{\delta \bar{\mathcal{P}}_A(t)}\,.
\end{aligned}
\tag{12.24}
$$

Since Ψ_H carries Grassmann signature and ghost number 1 and -1, respectively, the Grassmann signature and ghost number of a "star"-variable $\phi^\star$ is related to that of ϕ by

$$
\begin{aligned}
&\epsilon(\phi^\star) = \epsilon(\phi) + 1 \quad (mod\ 2)\,, \\
&gh(\phi^\star) = -gh(\phi) - 1\,.
\end{aligned}
$$

Thus in particular we have that $gh(\eta) = -gh(\bar{\mathcal{P}}) = 1$; $gh(q^\star) = gh(p^\star) = -1$; $gh(\eta^\star) = -2$; $gh(\bar{\mathcal{P}}^\star) = 0$.

Viewed as a function of the above variables, the FV-action - we denote it by S_H - takes the form

$$
\begin{aligned}
S_H[q,p,\eta,\bar{\mathcal{P}};q^\star,p^\star,\eta^\star,\bar{\mathcal{P}}^\star] = \int dt\Big(&\dot{q}^i p_i + \dot{\eta}^A\bar{\mathcal{P}}_A - H_{\mathcal{B}}(q,p,\eta,\bar{\mathcal{P}}) + q_i^\star\{q^i,Q_{\mathcal{B}}\} \\
&+p^{\star i}\{p_i,Q_{\mathcal{B}}\} + \eta_A^\star\{\eta^A,Q_{\mathcal{B}}\} + \bar{\mathcal{P}}^{\star A}\{\bar{\mathcal{P}}_A,Q_{\mathcal{B}}\}\Big)\,.
\end{aligned}
\tag{12.25}
$$

Let us denote by $\{\theta^\ell\}$ and $\{\theta_\ell^\star\}$ the sets $\{q^i,p_i,\eta^A,\bar{\mathcal{P}}_A\}$ and $\{q_i^\star,p^{\star i},\eta_A^\star,\bar{\mathcal{P}}^{\star A}\}$, and write S_H in the form

$$
S_H[q,p,\eta,\bar{\mathcal{P}};q^\star,p^\star,\eta^\star,\bar{\mathcal{P}}^\star] = \int dt\,\big(\dot{q}^i p_i + \dot{\eta}^A\bar{\mathcal{P}}_A - H_{\mathcal{B}}(q,p,\eta,\bar{\mathcal{P}})\big) + \int dt\, \theta_\ell^\star \hat{s}\theta^\ell
\tag{12.26}
$$

where $\hat{s}\theta^\ell$ is the BRST variation of θ^ℓ, with $\hat{s}f$ defined generically by

$$\hat{s}f = \{f, Q_{\mathcal{B}}\} \ , \tag{12.27}$$

where $f[\theta]$ is an arbitrary functional of θ. This definition follows quite naturally from the decomposition (12.26). [10] The ordering of the factors in the kinetic terms corresponds to the definiton of the canonical momenta in terms of left derivatives. Note also that the variables q^i include all variables (dynamical or not) of the original Lagrangian. Thus in the case of the Maxwell theory, it would include all components of A^μ and the corresponding canonically conjugate momenta. In a fixed gauge, i.e. for a given Ψ_H, the variables $\{\theta^\star_\ell\}$ are given functions of $\{\theta^\ell\}$. On the other hand, if the action (12.26) is regarded as a function of the independent variables q^i, p_i, η^A and $\bar{\mathcal{P}}_A$ and their respective antifields $q^\star_i$, $p^{\star i}$, $\eta^\star_A$ and $\bar{\mathcal{P}}^{\star A}$, then - as we now prove - (12.26) is found to satisfy the so-called classical master equation,

$$(S_H, S_H) = 0\,, \tag{12.28}$$

with the "antibracket" (f,g) defined as in (12.3) with $\vartheta^\ell \to \theta^\ell$, $\vartheta^\star_\ell \to \theta^\star_\ell$, and having the properties (12.5) and (12.6). Recall that ℓ labels the phase-space degrees of freedom, which in the case of fields will also include the spacial coordinates. In this case the sum over ℓ is understood to include an integration over the spacial coordinates.

The proof of (12.28) is based on the BRST invariance of the first integral in (12.26), and the Jacobi identity (12.6), implying $\{\{\Psi, Q_{\mathcal{B}}\}, Q_{\mathcal{B}}\} = 0$, where Ψ is an arbitrary function of the phase space variables [Rothe 2008].

Proof of the Hamiltonian master equation

Consider the antibracket

$$(S_H, S_H) = \int dt\ \left(\frac{\delta^{(r)} S_H}{\delta\theta^\ell(t)} \frac{\delta^{(l)} S_H}{\delta\theta^\star_\ell(t)} - \frac{\delta^{(r)} S_H}{\delta\theta^\star_\ell(t)} \frac{\delta^{(l)} S_H}{\delta\theta^\ell(t)} \right)\ . \tag{12.29}$$

Since S_H is a Grassmann even functional, and θ^ℓ and $\theta^\star_\ell$ have opposite Grassmann signature, it follows from (12.4) that

$$\frac{\delta^{(r)} S_H}{\delta\theta^\star_\ell} \frac{\delta^{(l)} S_H}{\delta\theta^\ell} = -\frac{\delta^{(r)} S_H}{\delta\theta^\ell} \frac{\delta^{(l)} S_H}{\partial\theta^\star_\ell}\ .$$

[10]This is the BRST variation as it is usually defined in the literature. The corresponding infinitesimal transformations are obtained by multiplying this expression by a global infinitesimal Grassmann valued parameter ϵ from the *right*. This is equivalent to the BRST variation $\delta_{\mathcal{B}} f = \epsilon\{Q_{\mathcal{B}}, f\}\epsilon$ introduced in chapter 11, where ϵ is multiplying the expression from the left.

Hence (12.29) takes the simpler form

$$(S_H, S_H) = 2\int dt\ \frac{\delta^{(r)} S_H}{\delta\theta^\ell(t)} \frac{\delta^{(l)} S_H}{\delta\theta^\star_\ell(t)}\ . \tag{12.30}$$

Let us now decompose (12.26) as follows:

$$S_H = S_{\mathcal{B}}[\theta] + \Delta[\theta, \theta^\star]\ ,$$

where

$$S_{\mathcal{B}} = \int dt\ \left(\dot{q}^i p_i + \dot{\eta}^A \bar{\mathcal{P}}_A - H_{\mathcal{B}}(q, p, \eta, \bar{\mathcal{P}})\right)$$

is the BRST invariant action, and

$$\Delta = \int dt\ \theta^\star_\ell \{\theta^\ell, Q_{\mathcal{B}}\} = \int dt\ \theta^\star_\ell \hat{s}\theta^\ell\ .$$

Hence (12.30) is given by

$$\begin{aligned}(S_H, S_H) &= 2\int dt \frac{\delta^{(r)} S_H}{\delta\theta^\ell(t)} \hat{s}\theta^\ell(t)\\ &= 2\int dt\ \left[\frac{\delta^{(r)} S_{\mathcal{B}}}{\delta\theta^\ell(t)} \hat{s}\theta^\ell(t) + \frac{\delta^{(r)} \Delta}{\delta\theta^\ell(t)} \hat{s}\theta^\ell(t)\right]\ .\end{aligned}$$

The first integral vanishes since $S_{\mathcal{B}}$ is BRST invariant. The second integral is just the BRST variation of Δ, with respect to θ^ℓ,

$$\int dt\ \theta^\star_\ell \hat{s}\{\theta^\ell, Q_{\mathcal{B}}\} = \int dt\ \theta^\star_\ell \{\{\theta^\ell, Q_{\mathcal{B}}\}, Q_{\mathcal{B}}\}\ ,$$

which, upon making use of the Jacobi identity, vanishes because of the nilpotency of $Q_{\mathcal{B}}$.

We have thus shown that the BRST invariance of $S_{\mathcal{B}}$ and the nilpotency of $Q_{\mathcal{B}}$ imply that the action S_H, considered as a function of the fields and antifields, satisfies the Hamiltonian master equation (12.28). For this to be the case, the *linear* dependence of S_H on the antifields has played an important role.

S_H can be regarded as the generator of a symmetry of the antifield-action in the antibracket sense. Indeed, from (12.26) and (12.27), we have

$$(\theta^\ell, S_H) = \frac{\delta^{(l)} S_H}{\delta\theta^\star_\ell} = \{\theta^\ell, Q_{\mathcal{B}}\}\ . \tag{12.31}$$

It thus follows, in particular, that the BRST invariant unitarized (gauge-fixed) action

$$S_U[\theta] = S_H[\theta, \frac{\delta\Psi_H}{\delta\theta}]$$

satisfies

$$(S_U, S_H) = 0 \,.$$

Hence S_U is invariant under transformations generated by S_H in the antibracket sense. Extending the variation (12.31) to include the antifields, we define a general variation induced by S_H on a functional f of θ^ℓ and $\theta^\star_\ell$ by

$$\overset{\vee}{s} f \equiv (f, S_H) \ . \tag{12.32}$$

In particular we have that

$$\overset{\vee}{s} \theta^\star_\ell = (\theta^\star_\ell, S_H) \ . \tag{12.33}$$

From (12.32) and (12.28) we then conclude that the complete set of transformations are also a symmetry of S_H.

Example

Consider again our quantum mechanical model defined by the action (12.18), with primary and secondary first class constraints $\phi = p_y$ and $T = x - p_x$, respectively. The BRST charge is given by

$$Q_{\mathcal{B}} = G_A \eta^A = p_y \eta^1 + (x - p_x)\eta^2 \ .$$

From (12.21) and (12.22) we then obtain [11]

$$\{\Psi, Q_{\mathcal{B}}\} = -x^\star \eta^2 + y^\star \eta^1 - p_x^\star \eta^2 - \bar{\mathcal{P}}_1^\star p_y - \bar{\mathcal{P}}_2^\star (x - p_x) \,.$$

Hence S_H in (12.26) takes the form

$$\begin{aligned} S_H = &\int dt \left(\frac{1}{2}[(p_x \dot{x} - x \dot{p}_x) + (p_y \dot{y} - y \dot{p}_y)] + \dot{\eta}^1 \bar{\mathcal{P}}_1 + \dot{\eta}^2 \bar{\mathcal{P}}_2 - H_{\mathcal{B}} \right) \\ &+ \int dt \, \left(-x^\star \eta^2 + y^\star \eta^1 - p_x^\star \eta^2 - \bar{\mathcal{P}}_1^\star p_y - \bar{\mathcal{P}}_2^\star (x - p_x) \right) \ , \end{aligned} \tag{12.34}$$

where

$$H_{\mathcal{B}} = \frac{1}{2}(p_x^2 - x^2) + \eta^2 \bar{\mathcal{P}}_2$$

is the BRST Hamiltonian. One now checks, making use of (12.5), that the Hamiltonian master equation is satisfied. Indeed, as expected from similar

[11] Recall that $\{\eta^A, \bar{\mathcal{P}}_B\} = -\delta^A_B$.

considerations in earlier chapters, the *kinetic* part of the action (12.34) is found to be invariant under the transformations (12.32):

$$
\begin{aligned}
\overset{\vee}{s} x &= -\eta^2 \,, & \overset{\vee}{s} y &= \eta^1 \,, \\
\overset{\vee}{s} p_x &= -\eta^2 \,, & \overset{\vee}{s} p_y &= 0 \,, \\
\overset{\vee}{s} \eta^1 &= 0 \,, & \overset{\vee}{s} \eta^2 &= 0 \,, \\
\overset{\vee}{s} \bar{\mathcal{P}}_1 &= -p_y \,, & \overset{\vee}{s} \bar{\mathcal{P}}_2 &= p_x - x \,.
\end{aligned}
$$

Explicitly one finds

$$
\overset{\vee}{s}_H \Big(\frac{1}{2}[(p_x \dot{x} - x\dot{p}_x) + (p_y \dot{y} - y \dot{p}_y)] + \dot{\eta}^1 \bar{\mathcal{P}}_1 + \dot{\eta}^2 \bar{\mathcal{P}}_2 \Big) = -\frac{1}{2}\frac{dQ_{\mathcal{B}}}{dt} \,.
$$

The same is true for the last group of terms in (12.34), after making use of the additional transformation laws

$$
\begin{aligned}
\overset{\vee}{s} \bar{\mathcal{P}}_1^\star &= \dot{\eta}^1 \,, \\
\overset{\vee}{s} \bar{\mathcal{P}}_2^\star &= \dot{\eta}^2 - \eta^2 \,, \\
\overset{\vee}{s} x^\star &= \dot{p}_x - x + \bar{\mathcal{P}}_2^\star \,, \\
\overset{\vee}{s} y^\star &= \dot{p}_y \,, \\
\overset{\vee}{s} p_x^\star &= = -\dot{x} + p_x - \bar{\mathcal{P}}_2^\star \,, \\
\overset{\vee}{s} p_y^\star &= -\dot{y} + \bar{\mathcal{P}}_1^\star \,,
\end{aligned}
$$

following from (12.33). One finds

$$
\overset{\vee}{s} \Big(- x^\star \eta^2 + y^\star \eta^1 - p_x^\star \eta^2 - \bar{\mathcal{P}}_1^\star p_y - \bar{\mathcal{P}}_2^\star (x - p_x) \Big) = -\frac{dQ_{\mathcal{B}}}{dt} \,,
$$

as well as

$$
\overset{\vee}{s} H_{\mathcal{B}} = 0 \,.
$$

One checks that the above transformation laws are nilpotent as expected. The change in the action induced by the complete set of variations is thus given by

$$
\overset{\vee}{s} S = -\frac{3}{2} \int dt \; \frac{dQ_{\mathcal{B}}}{dt}
$$

so that indeed $(S_H, S_H) = 0$. Hence (12.34) is a solution to the Hamiltonian master equation.

12.3.2 Transition to configuration space

We now come to the second part of our program mentioned at the end of section 1, i.e., establishing the Lagrangian counterpart to the Hamiltonian master equation (12.28). We expect that the Lagrangian BRST symmetry is again generated by some nilpotent operator, and that the symmetry transformations in configuration space are just the "pullbacks" of the BRST transformations in phase space.

Consider the FV phase space partition function (11.103) associated with the action (12.25) in a fixed gauge, with the star-variables given by (12.24),

$$Z_\Psi = \int DqDpD\eta D\bar{\mathcal{P}} \int Dq^\star Dp^\star D\eta^\star D\bar{\mathcal{P}}^\star \prod_{i,t} \delta\left(q_i^\star - \frac{\delta\Psi_H}{\delta q_i}\right) \delta\left(p^{\star i} - \frac{\delta\Psi_H}{\delta p_i}\right)$$
$$\times \prod_{A,t} \delta\left(\eta_A^\star - \frac{\delta\Psi_H}{\delta\eta^A}\right) \delta\left(\bar{\mathcal{P}}^{\star A} - \frac{\delta\Psi_H}{\delta\bar{\mathcal{P}}^A}\right) e^{iS_H[q,\eta,\bar{\mathcal{P}};\; q^\star,\eta^\star,\bar{\mathcal{P}}^\star]}\,, \tag{12.35}$$

where S_H is given by (12.26). As already pointed out, the transition to the Lagrangian formulation cannot be effected in the general case. Assumptions must be made, which, however, include many cases of physical interest. The class of systems we shall consider are *gauge theories of rank one with only first class constraints, where each primary constraint gives rise to just one secondary constraint.* Hence the number of gauge identities, and therefore also gauge parameters, equals the number of primaries (or secondaries). The $SU(N)$ Yang-Mills theory is of this type. Our example in section 5 of the previous chapter has shown however, that this is not necessarily a "must".

To keep the discussion as simple and transparent as possible, we will for the moment consider systems with a finite number of degrees of freedom. Suppose we are given the classical Hamiltonian $H(q,p)$ and the first class constraints G_A. The primary constraints we denote by ϕ_α; they are assumed each to give rise to just one secondary (first class) constraint T_α. In the following it is convenient to define the latter in the strong sense by

$$\{H,\phi_\alpha\} = T_\alpha \quad (\textit{secondary constraints})\ . \tag{12.36}$$

For a gauge theory of rank *one* the BRST charge and Hamiltonian are given by (see chapter 11)

$$Q_{\mathcal{B}} = G_A\eta^A + \frac{1}{2}\bar{\mathcal{P}}_A U^A_{BC}\eta^B\eta^C \tag{12.37}$$

and

$$H_{\mathcal{B}}(q,p,\eta,\bar{\mathcal{P}}) = H(q,p) + \eta^A V_A{}^B\bar{\mathcal{P}}_B\ , \tag{12.38}$$

with $G_A = \{\phi_\alpha, T_\alpha\}$ the first class constraints, and $V_A{}^B, U^C_{AB}$ the corresponding structure functions defined by (11.105).

In order to make the transition to configuration space, which requires us to perform all momentum integrations, we will make a number of assumptions, which can however be sometimes relaxed in some explicit models:

i) The constraints $G_A(q,p)$ are at most linear in the momenta p_i.

ii) Each primary constraint gives rise to a secondary constraint according to (12.36), and the algebra of the secondaries T_α with themselves and with H, closes on itself,

$$\{H, T_\alpha\} = h_\alpha{}^\beta T_\beta \ , \quad \{T_\alpha, T_\beta\} = f^\gamma_{\alpha\beta} T_\gamma \tag{12.39}$$

with structure functions which do not depend on the momenta p_i. [12]

iii) The "fermion gauge fixing functional" Ψ_H is independent of the momenta p_i, and the momenta P^α and $\bar{P}_\alpha$ defined in (11.122); i.e.,

$$\Psi_H = \Psi_H[q, c, \bar{c}].$$

Note also that *iii)* is not really a fundamental restriction, since we are allowed to evaluate BRST invariant quantities in any gauge, i.e. for any choice of Ψ_H. Hence *iii)* is just a convenient choice of gauge.

An immediate consequence of the above choice for Ψ_H is that

$$p_i^\star = P_\alpha^\star = \bar{P}_\alpha^\star = 0 \tag{12.40}$$

and as a consequence of assumption *ii)* we have from (12.38) that

$$H_{\mathcal{B}} = H + P^\alpha \bar{P}_\alpha + c^\alpha h_\alpha{}^\beta \bar{P}_\beta \, .$$

We remark here that the $P^\alpha \bar{P}_\alpha$-term, whose presence follows immediately from the definition of the secondary constraints (12.36), is crucial for what follows.

The following assumption simplifies the presentation but is not essential. We assume that the primary constraints are of the form [13]

$$\phi_\alpha \equiv p_\alpha = 0 \ \ ; \ \ (\alpha = 1, \ldots, N) \, ,$$

and that the secondary constraints T_α are linear in the momenta, but do not depend on the p_α's. This is the case if we choose H to be the canonical Hamiltonian evaluated on the primary surface. Then the constraints, T_α, defined in

[12] As we have seen in chapter 11, the structure functions of the pure Maxwell and Yang-Mills theories satisfy these requirements.

[13] If the Lagrangian depends linearly on the velocities $\dot{q}_\alpha$, then this can always be achieved by a redefinition of variables.

(12.36), do not depend on the p_α's, With this asumption, and the restrictions *ii*) above, the BRST charge (12.37) is given by

$$Q_{\mathcal{B}} = P^\alpha p_\alpha + c^\alpha T_\alpha + \frac{1}{2} f^{\gamma}_{\ \alpha\beta} \bar{P}_\gamma c^\alpha c^\beta \ . \tag{12.41}$$

Furthermore, making use of the definitions (11.122), the partition function (12.35) takes the explicit form

$$\begin{aligned} Z_\Psi = & \int DqDpDcD\bar{c}DPD\bar{P} \int Dq^\star Dc^\star D\bar{c}^\star \prod_{\alpha,t} \delta\left(c^\star_\alpha - \frac{\delta\Psi_H}{\delta c^\alpha}\right) \delta\left(\bar{c}^{\star\alpha} - \frac{\delta\Psi_H}{\delta\bar{c}_\alpha}\right) \\ & \times \prod_{i,t} \delta\left(q^\star_i - \frac{\delta\Psi_H}{\delta q_i}\right) e^{i\int dt\ \left[\dot{q}^i p_i + \dot{P}^\alpha \bar{c}_\alpha + \dot{c}^\alpha \bar{P}_\alpha - (H + P^\alpha \bar{P}_\alpha + c^\alpha h_\alpha^{\ \beta} \bar{P}_\beta)\right]} \\ & \times e^{i\int dt [q^\star_i \hat{s} q^i + c^\star_\alpha \hat{s} c^\alpha + \bar{c}^{\star\alpha} \hat{s}\bar{c}_\alpha]} \ , \end{aligned} \tag{12.42}$$

where we have implemented (12.40), and where from (12.27) and (12.41) we have that

$$\begin{aligned} \hat{s} q^\alpha &= P^\alpha \ , \\ \hat{s} q^a &= c^\alpha \frac{\partial T_\alpha}{\partial p_a} \ , \\ \hat{s} c^\alpha &= -\frac{1}{2} f^\alpha_{\beta\gamma} c^\beta c^\gamma \ , \\ \hat{s} \bar{c}_\alpha &= -p_\alpha \ . \end{aligned} \tag{12.43}$$

Since with our above assumptions these expressions do not depend on p_a and $\bar{P}_\alpha$, [14] we can perform the integration over $\bar{P}_\alpha$ in (12.42). This yields a δ-function which fixes P^α as a function of the coordinates and their time derivatives (pull-back of the Legendre transformation):

$$P^\alpha \to D^\alpha_{0\beta} c^\beta \ , \tag{12.44}$$

where

$$D^\alpha_{0\beta}(q) = \delta^\alpha_\beta \partial_0 + \bar{h}^\alpha_{\ \beta}(q)$$

and

$$\bar{h}^\alpha_{\ \beta}(q) = h_\beta^{\ \alpha}(q) \ , \tag{12.45}$$

with $h_\alpha^{\ \beta}$ defined in (12.39). Hence from (12.44) and (12.43)

$$\hat{s} q^\alpha \to D^\alpha_{0\beta} c^\beta \ . \tag{12.46}$$

[14] Recall that T_α was assumed to depend at most linearly on the momenta p_a.

After performing also the integration over the momenta p_a, we are then left with the following expression for the partition function in configuration space

$$
\begin{aligned}
Z_\Psi^L = \int \prod_\alpha Dp_\alpha \int Dq D\bar{c} Dc \int Dq^\star D\bar{c}^\star Dc^\star \prod_{i,t} \delta\left(q_i^\star - \frac{\delta \Psi_H}{\delta q_i}\right) \\
\times \prod_{\alpha,t} \delta\left(c_\alpha^\star - \frac{\delta \Psi_H}{\delta c^\alpha}\right) \times \prod_{\alpha,t} \delta\left(\bar{c}_\alpha^\star - \frac{\delta \Psi_H}{\delta \bar{c}_\alpha}\right) e^{iS_L[q,c,\bar{c};q^\star,c^\star,\bar{c}^\star;p_\alpha]} \quad (12.47)
\end{aligned}
$$

where [15]

$$
\begin{aligned}
S_L[q,c,\bar{c};q^\star,c^\star,\bar{c}^\star] = \int dt\, \Big(L(q,\dot{q}) + (q_\alpha^\star + \dot{\bar{c}}_\alpha) \overset{\vee}{s}\, q^\alpha + q_a^\star \overset{\vee}{s}\, q^a + c_\alpha^\star \overset{\vee}{s}\, c^\alpha \\
- (\bar{c}^{\star\alpha} - \dot{q}^\alpha) p_\alpha \Big) \quad (12.48)
\end{aligned}
$$

and $\overset{\vee}{s}\theta^\ell$ is the "pullback" of $\hat{s}\theta^\ell$ to the configuration space level:

$$
\begin{aligned}
\overset{\vee}{s} q^a &= \hat{s} q^a \ , \\
\overset{\vee}{s} q^\alpha &= D_{0\beta}^\alpha c^\beta \ , \\
\overset{\vee}{s} c^\alpha &= \hat{s} c^\alpha \ , \\
\overset{\vee}{s} \bar{c}_\alpha &= \hat{s} \bar{c}_\alpha \ .
\end{aligned} \quad (12.49)
$$

$L(q, \dot{q})$ is the classical Lagrangian. Let us now introduce the following "Lagrangian fermion gauge fixing functional",

$$
\Psi_L[q,c,\bar{c}] = \Psi_H[q,c,\bar{c}] - \int dt\ \bar{c}_\alpha \dot{q}^\alpha \ , \quad (12.50)
$$

in accordance with our assumption that Ψ_H does not depend on the momenta. Then (12.47) can be written in the form

$$
\begin{aligned}
Z_\Psi^L = \int DB \int Dq Dc D\bar{c} \int Dq^\star Dc^\star D\bar{c}^\star \prod_{i,t} \delta\left(q_i^\star - \frac{\delta \Psi_L}{\delta q_i}\right) \prod_{\alpha,t} \delta\left(c_\alpha^\star - \frac{\delta \Psi_L}{\delta c_\alpha}\right) \\
\times \prod_{\alpha,t} \delta\left(\bar{c}^{\star\alpha} - \frac{\delta \Psi_L}{\delta \bar{c}_\alpha}\right) e^{i\int dt\, S_L[q,c,\bar{c};q^\star,c^\star,\bar{c}^\star;B]} , \quad (12.51)
\end{aligned}
$$

with

$$
S[q,c,\bar{c};q^\star,c^\star,\bar{c}^\star;B] = S_{cl}[q] + \int dt\, \left(q_i^\star \overset{\vee}{s} q^i + c_\alpha^\star \overset{\vee}{s} c^\alpha + \bar{c}^{\star\alpha} B_\alpha \right) \ , \quad (12.52)
$$

[15] The momentum p_α in (12.48) plays the role of $-B_\alpha$ in (12.16). We have left it in this form, in order to emphasize the meaning of the fields B_α appearing in the literature.

where we have now set $p_\alpha = -B_\alpha$ in order to follow the notation in the literature, and where the antifields are now defined in terms of Ψ_L as in (12.17): [16]

$$\vartheta^\star_\ell = \frac{\delta \Psi_L[\vartheta]}{\delta \vartheta^\ell} \,. \tag{12.53}$$

For a given functional $\Psi_L[\vartheta]$ this corresponds to fixing the gauge [Batalin 1981]. Note that from the structure of (12.52) it follows trivially that

$$\overset{\vee}{s} q^i = (q^i, S) \;, \quad \overset{\vee}{s} c^\alpha = (c^\alpha, S) \;,$$

where the bracket (f,g), with f and g functions of θ^ℓ and $\theta\star_\ell$, is defined in (12.3).

Upon carrying out the integrations over the antifields in (12.51), these become fixed functions of q_i, c^α, and $\bar{c}_\alpha$. Choosing Ψ_H in (12.50) to be

$$\Psi_H = \int dt \; \bar{c}_\alpha \chi^\alpha(q) \,, \tag{12.54}$$

we have

$$q^\star_a = \bar{c}_\beta \frac{\partial \chi^\beta}{\partial q^a} \;, \quad q^\star_\alpha = \bar{c}_\beta \frac{\partial \chi^\beta}{\partial q^\alpha} + \dot{\bar{c}}_\alpha \;,$$
$$c^\star_\alpha = 0 \;, \quad \bar{c}^{\star\alpha} = \chi^\alpha - \dot{q}^\alpha \;,$$

and finally obtain

$$Z^L_\Psi = \int DB \int DqDcD\bar{c} \; e^{i\int dt \; \left(L(q,\dot{q}) - \bar{c}_\alpha \partial^0 D^\alpha_{0\beta} c^\beta + \bar{c}_\alpha \{\chi^\alpha, T_\beta\} c^\beta + (\chi^\alpha - \dot{q}^\alpha) B_\alpha\right)} \;.$$

Note that the B_α's are the Lagrange multipliers implementing the dynamical gauge conditions,

$$\dot{q}^\alpha - \chi^\alpha = 0 \;,$$

which involves the time derivative of q^α.

Under the various assumptions stated above, we see that, analogous to the Hamiltonian action S_H, the Lagrangian action (12.52) is a *linear* function of the antifields and of the form (12.16). In general, the action may contain higher orders in the antifields.

Notice that for the above class of gauges $c^\star_\alpha = 0$, so that $\overset{\vee}{s} c^\alpha$ does not enter, and only knowledge of the symmetry of the *classical* action is required for constructing the gauge fixed partition function Z^L_Ψ.

The following example shows that the requirements *i)* and *ii)* need not necessarily be satisfied for obtaining the representation (12.51).

[16] We have also dispensed of a subscript "L" on S in order to follow the notation in the literature.

Example: The Nambu-Goto model revisited

In the following we reconsider the Nambu-Goto model of the bosonic string in the form of the action given in (5.34) where, for convenience, we set $\lambda = \lambda^1$ and $\mu = \lambda^2$:

$$S = \int d\sigma \left(\frac{1}{2}\frac{\dot{x}^2}{\lambda^1} - \frac{\lambda^2}{\lambda^1}\dot{x}x' + \frac{1}{2}\frac{(\lambda^2)^2}{\lambda^1}x'^2 - \frac{1}{2}\lambda^1 x'^2 \right) .$$

As was seen in chapter 5, it is a first class theory with two primary and two secondary constraints, of which one of the secondaries involves the phase space momentum quadratically. Hence the model falls *outside* the class of models we have considered so far. As we shall demonstrate, the unitarized Hamiltonian nevertheless leads to a partition function in configuration space which is of the BV form. [17]

We denote the primary constraints by $\phi_\alpha = 0$ $(\alpha = 1, 2)$. Then

$$\phi_\alpha = \pi_\alpha \ , \quad (\alpha = 1, 2) \ ,$$

where π_α are the momenta conjugate to the fields λ^α. The two secondary constraints read

$$T_1 := \frac{1}{2}(p^2 + x'^2) = 0 \ , \quad T_2 := p \cdot x' = 0 \ ,$$

where $x^\mu = x^\mu(\sigma, \tau)$ and $x'^\mu = \partial_\sigma x^\mu$. In terms of these the canonical Hamiltonian on Γ_P reads (cf. (5.37))

$$H_0 = \int d\sigma \lambda^\alpha T_\alpha \, ,$$

which is a "zero" Hamiltonian, characteristic of reparametrization invariant theories. The secondary constraints satisfy the closed algebra (see (5.36))

$$\begin{aligned}
\{T_1(\sigma), T_1(\sigma')\} &= T_2(\sigma)\partial_\sigma \delta(\sigma - \sigma') - T_2(\sigma')\partial_{\sigma'}\delta(\sigma - \sigma') \ , \\
\{T_1(\sigma), T_2(\sigma')\} &= T_1(\sigma)\partial_\sigma \delta(\sigma - \sigma') - T_1(\sigma')\partial_{\sigma'}\delta(\sigma - \sigma') \ , \\
\{T_2(\sigma), T_2(\sigma')\} &= T_2(\sigma)\partial_\sigma \delta(\sigma - \sigma') - T_2(\sigma')\partial_{\sigma'}\delta(\sigma - \sigma') \ .
\end{aligned}$$

This algebra can be written in the form (11.97),

$$\{T_\alpha(\sigma), T_\beta(\sigma')\} = \int d\sigma'' \ u^\gamma_{\alpha\beta}(\sigma, \sigma'; \sigma'')T_\gamma(\sigma'')$$

[17] In the following we adapt the notation to that of chapter 11.

where

$$
\begin{aligned}
u_{11}^2(\sigma,\sigma',\sigma'') &= \delta(\sigma''-\sigma)\partial_\sigma\delta(\sigma-\sigma') - \delta(\sigma''-\sigma')\partial_{\sigma'}\delta(\sigma'-\sigma)\,,\\
u_{12}^1(\sigma,\sigma',\sigma'') &= \delta(\sigma''-\sigma)\partial_\sigma\delta(\sigma-\sigma') - \delta(\sigma''-\sigma')\partial_{\sigma'}\delta(\sigma'-\sigma) \quad (12.55)\\
u_{22}^2(\sigma,\sigma',\sigma'') &= \delta(\sigma''-\sigma)\partial_\sigma\delta(\sigma-\sigma') - \delta(\sigma''-\sigma')\partial_{\sigma'}\delta(\sigma'-\sigma)\,,
\end{aligned}
$$

with $u_{21}^1(\sigma',\sigma;\sigma'') = -u_{12}^1(\sigma,\sigma';\sigma'')$. The primary constraints have vanishing Poisson brackets with all constraints.

The structure functions V_A^B defined in (11.100) can be read off from the Poisson brackets written in the form

$$
\{H_0,\phi_\alpha(\sigma)\} = \int d\sigma\ \delta_\alpha^{\ \beta}\delta(\sigma-\sigma')T_\alpha(\sigma')\ , \tag{12.56}
$$

$$
\{H_0,T_\alpha(\sigma)\} = \int d\sigma\ v_\alpha^\beta(\sigma,\sigma')T_\beta(\sigma')\ .
$$

A simple calculation yields

$$
\begin{aligned}
\{H_0,T_1\} &= -(\lambda^1\partial_\sigma T_2 + 2\partial_\sigma\lambda^1 T_2 + \lambda^2\partial_\sigma T_1 + 2\partial_\sigma\lambda^2 T_1)\ ,\\
\{H_0,T_2\} &= -(\lambda^1\partial_\sigma T_1 + 2\partial_\sigma\lambda^1 T_1 + \lambda^2\partial_\sigma T_2 + 2\partial_\sigma\lambda^2 T_2)\ .
\end{aligned}
$$

Hence

$$
\begin{aligned}
v_1^1(\sigma,\sigma') = v_2^2(\sigma,\sigma') &= -\left(\lambda^2(\sigma)\partial_\sigma + 2\partial_\sigma\lambda^2(\sigma)\right)\delta(\sigma-\sigma')\\
v_1^2(\sigma,\sigma') = v_2^1(\sigma,\sigma') &= -\left(\lambda^1(\sigma)\partial_\sigma + 2\partial_\sigma\lambda^1(\sigma)\right)\delta(\sigma-\sigma')\,.
\end{aligned}
$$

Note, that the derivatives appearing in the structure functions do *not* reflect a momentum dependence of the structure functions. Hence the structure functions satisfy the restrictive conditions ii) of the last section. In fact, we have a theory of rank *one*, and therefore have for the BRST charge (see (11.98)),

$$
\begin{aligned}
Q_{\mathcal{B}} &= \int \left(P^\alpha\phi_\alpha + c^\alpha T_\alpha + \frac{1}{2}\bar{P}_\gamma u^\gamma_{\ \alpha\beta}c^\alpha c^\beta\right)\\
&= \int d\sigma\left(P^\alpha\pi_\alpha + c^\alpha T_\alpha + \bar{P}_1(c^1\partial_\sigma c^2 + c^2\partial_\sigma c^1) + \bar{P}_2(c^1\partial_\sigma c^1 + c^2\partial_\sigma c^2)\right)\ ,
\end{aligned}
$$

where in the first expression the integral sign stands for a single or multiple integral, as the case may be, and we have identified ϕ_α with π_α in the second integral.

Assuming the Hamiltonian gauge fixing fermion to be of the form

$$
\Psi_H = \int d\sigma d\tau\ \bar{c}_\alpha\chi^\alpha(x,\lambda)\ , \tag{12.57}
$$

we have that $\pi^\star = p^\star_\alpha = c^\star_\alpha = P^\star_\alpha = \bar{P}^{\star\alpha} = 0$, and the unitarized Hamiltonian (11.102) reads,

$$H_U = \int d\sigma \lambda^\alpha T_\alpha - \int d\sigma \ \big[(\partial_\sigma \lambda^2 c^1 - \lambda^2 \partial_\sigma c^1 + \partial_\sigma \lambda^1 c^2 + \lambda^1 \partial_\sigma c^2)\bar{P}_1$$
$$+ (\partial_\sigma \lambda^2 c^2 - \lambda^2 \partial_\sigma c^2 + \partial_\sigma \lambda^1 c^1 - \lambda^1 \partial_\sigma c^1)\bar{P}_2\big] + \int d\sigma \ P^\alpha \bar{P}_\alpha$$
$$- \int d\sigma [x^\star_\mu \hat{s} x^\mu + \lambda^\star_\alpha \hat{s}\lambda^\alpha + \bar{c}^{\star\alpha}\hat{s}\bar{c}_\alpha] \ , \tag{12.58}$$

where the star-variables have been defined in (12.24) in terms of the *Hamiltonian* gauge fixing fermion functional Ψ_H, and where we have for the BRST transformations generated by $Q_{\mathcal{B}}$,

$$\hat{s}x^\mu \equiv \{x^\mu, Q_{\mathcal{B}}\} = c^1 p^\mu + c^2 \partial_\sigma x^\mu \ , \tag{12.59}$$
$$\hat{s}\bar{c}_\alpha = \{\bar{c}_\alpha, Q_{\mathcal{B}}\} = -\pi_\alpha \ ,$$

and

$$\hat{s}\lambda^\alpha = P^\alpha \, . \tag{12.60}$$

Note that we have made manifest the $P\bar{P}$ term arising from the Poisson bracket (12.56). It plays a central role to get the transformation laws (12.60) for the fields λ^α in configuration space. Indeed, performing the $\bar{P}_\alpha$ integration in the functional integral,

$$Z = \int DxDpD\lambda D\pi \int DcD\bar{c}DPD\bar{P} \ e^{i\int d\sigma d\tau [\dot{x}\cdot p + \dot{\lambda}^\alpha \pi_\alpha + \dot{c}^\alpha \bar{P}_\alpha + \dot{\bar{c}}^\alpha P_\alpha - \mathcal{H}_U]} \tag{12.61}$$

we are led to the delta functions,

$$\delta[P^1 - (\partial_\tau c^1 - \lambda^2 \partial_\sigma c^1 + \partial_\sigma \lambda^2 c^1 - \lambda^1 \partial_\sigma c^2 + \partial_\sigma \lambda^1 c^2)] \, ,$$
$$\delta[P^2 - (\partial_\tau c^2 - \lambda^2 \partial_\sigma c^2 + \partial_\sigma \lambda^2 c^2 - \lambda^1 \partial_\sigma c^1 + \partial_\sigma \lambda^1 c^1)] \, .$$

Doing next the P^α integrations we thus recover with (12.60) the transformation laws (5.39) and (5.40), with α^β replaced by c^β and p_μ identified with the momentum canonically conjugate to x^μ:

$$\overset{\vee}{s}\lambda^1 = (\partial_\tau c^1 - \lambda^2 \partial_\sigma c^1 + \partial_\sigma \lambda^2 c^1 - \lambda \partial_\sigma c^2 + \partial_\sigma \lambda^1 c^2) \, ,$$
$$\overset{\vee}{s}\lambda^2 = (\partial_\tau c^2 - \lambda^2 \partial_\sigma c^2 + \partial_\sigma \lambda^2 c^2 - \lambda \partial_\sigma c^1 + \partial_\sigma \lambda^1 c^1) \ . \tag{12.62}$$

Joining the $\dot{\lambda}^\alpha \pi_\alpha$ term in (12.61) to $\bar{c}^{\star\alpha}\hat{s}\bar{c}_\alpha$ in (12.58), and making use of (12.60) and $\hat{s}\bar{c}_\alpha = -\pi_\alpha$ one is then led to the partition function

$$Z = \int DxDpD\lambda D\bar{c}Dc \ e^{i\int d\sigma d\tau [\dot{x}\cdot p - \frac{\lambda^1}{2}(p^2 + x'^2) - \lambda^2 p\cdot x']}$$
$$\times \, e^{i\int d\sigma d\tau \ [x^\star_\mu \hat{s} x^\mu + (\lambda^\star_\alpha + \dot{\bar{c}}_\alpha)\hat{s}\lambda^\alpha - (\bar{c}^{\star\alpha} - \dot{\lambda}^\alpha)\pi_\alpha]} \, .$$

Introducing the Lagrangian gauge fixing fermion

$$\Psi_L = \Psi_H - \int d\sigma d\tau \ \bar{c}_\alpha \dot{\lambda}^\alpha \ , \tag{12.63}$$

where Ψ_H is given by (12.57), the above partition function can be put into the form

$$Z = \int DxDpD\lambda D\pi \int D\bar{c}Dc \ e^{i\int d\sigma d\tau [\dot{x}\cdot p - \frac{\lambda^1}{2}(p^2+x'^2) - \lambda^2 p\cdot x']}$$
$$\times \ e^{i\int \ d\sigma d\tau \ [x^\star_\mu \hat{s}x^\mu + \lambda^\star \hat{s}\lambda^\alpha - \bar{c}^{\star\alpha}\pi_\alpha]} ,$$

where the antifields, now labeled by an asterix, are computed with respect to the Lagrangian gauge-fixing function.

We now assume χ not to depend on π_α and the ghosts. The π-integration then yields the delta function $\prod_\alpha \delta[\dot{\lambda}^\alpha - \chi^\alpha]$. In order to perform the p integration one must take account of the p dependence of $\hat{s}x^\mu$ (cf. eq. (12.59)). One finds

$$Z = \int DxD\lambda DcD\bar{c} \prod_{x,\alpha=1}^{2} \delta[\dot{\lambda}^\alpha - \chi^\alpha] e^{iS} \ ,$$

where

$$S = S_{NG} + \int d\sigma d\tau \ [x^\star_\mu \overset{\vee}{s} x^\mu + \lambda^\star_\alpha \overset{\vee}{s} \lambda^\alpha] \, ,$$

and

$$\overset{\vee}{s} x^\mu = c^1 \left[\frac{1}{\lambda^1}(\partial_\tau x^\mu) - \frac{\lambda^2}{\lambda_1}(\partial_\sigma x^\mu) \right] + c^2(\partial_\sigma x^\mu) \tag{12.64}$$

is the "pullback" of the x^μ-symmetry transformation for the Nambu-Goto action, which in phase space is given by (12.59). Here we have made use of the Grassmann character of the ghosts to set $(c^1)^2 = 0$, in order to drop a term *quadratic* in $x^\star_\mu$. We conclude that we have recovered the BV form of the partition function. This is remarkable, since this time the secondary constraint T_1 is quadratic in the momenta, and therefore falls outside of our asumptions leading from the phase-space FV partition function to the final form (12.51) of the BV partition function.

In order to make contact with the Faddeev-Popov method, we must perform the remaining ghost integrations, taking account of the ghost dependence of $\lambda^\star_\alpha$ and $x^\star_\mu$. Making use of (12.64) and (12.62) we can write

$$x^\star_\mu \ \overset{\vee}{s} \ x^\mu + \lambda^\star_\alpha \ \overset{\vee}{s} \ \lambda^\alpha = \bar{c}_\alpha M^\alpha_{\ \beta} c^\beta \ ,$$

where M is calculated in the gauge (12.63). We leave it to the reader to write down the explicit form for M^α_β. Our final result thus reads

$$Z = \int DxD\lambda \prod_{x,\alpha=1}^{2} \delta(\dot{\lambda}^\alpha - \chi^\alpha) \det M \; e^{iS_{NG}} \;,$$

where $\det M$ is the Faddeev-Popov determinant associated with the symmetry transformations (12.62) and (12.64) in configurations space in the gauge $\dot{\lambda}^\alpha - \chi(x,\lambda) = 0$.

Elevating the master equation to an axiomatic level, one can now seek solutions also for models that do not satisfy our assumptions. In that case higher order terms in the antifields may turn out to be present.

12.4 The Lagrangian master equation

In the previous section we have seen that on Hamiltonian level the action (12.26) considered as a function of the phase space variables q, p, η, $\mathcal{P}$ and their "anti"-partners, satisfies the master equation (12.28), without any restrictions on the structure functions V_A^B, U_{BC}^A, and fermion gauge fixing function Ψ_H. This is a consequence of the linear dependence of the action (12.25) on the antifields. Furthermore, the transformations of the phase space variables generated by the BRST charge left the gauge-fixed phase space action invariant, irrespective of the choice of gauge fixing functional contained in a BRST exact term.

In what follows we shall drop the subscript on S_L in order to conform to the notation in the literature. On the configuration space level we again consider two types of actions: the field-antifield action $S[\vartheta, \vartheta^\star]$, where $\vartheta := (q, c, \bar{c})$, and the gauged fixed action

$$S_{gf}[\vartheta] = S[\vartheta, \frac{\delta\Psi_L}{\delta\vartheta}]. \tag{12.65}$$

Motivated by our Hamiltonian analysis in the last section and the form of the action (12.52), we are led to introduce the Lagrangian antibracket (12.3) and the following variation of a functional $F[\vartheta, \vartheta^\star]$,

$$\overset{\vee}{s} F = (F, S) \;. \tag{12.66}$$

It then follows that the variation of S vanishes if S satisfies the master equation

$$(S, S) = 0 \;, \tag{12.67}$$

which is the analog of (12.28). Furthermore it follows - after making use of the Jacobi identity - that $\overset{\vee}{s}$ is nilpotent, i.e.

$$\overset{\vee}{s}{}^2 F = ((F, S), S) = 0 \;.$$

Of particular interest are nilpotent transformations which leave the *gauge fixed* action (12.65) invariant. As we shall show below, they are given by

$$\overset{\vee}{s}_\Psi \vartheta^\ell = (\vartheta^\ell, S)_{gf} = \left(\frac{\partial^{(l)} S}{\partial \vartheta^\star_\ell}\right)_{gf} \ , \tag{12.68}$$

where the subscript "gf" stands for "gauge fixed"; i.e. the bracket is to be evaluated at $\vartheta^\star_\ell = \frac{\partial \Psi_L}{\partial \vartheta^\ell}$. Since

$$\overset{\vee}{s}_\Psi f[\vartheta] = \int dt \frac{\delta^{(r)} f}{\delta \vartheta^\ell} \overset{\vee}{s}_\Psi \vartheta^\ell \ ,$$

eq. (12.68) generalizes to

$$\overset{\vee}{s}_\Psi f[\vartheta] = (f, S)_{gf} \ . \tag{12.69}$$

We next verify that S_{gf} is indeed invariant under the transformations (12.68), if S is a solution to the master equation.

Consider the case where f in (12.69) is given by S_{gf}. Then

$$(S_{gf}, S)_{gf} = \int dt \ \frac{\delta^{(r)} S_{gf}}{\delta \vartheta^\ell(t)} \left(\frac{\delta^{(l)} S}{\delta \vartheta^\star_\ell(t)}\right)_{gf} \ .$$

Now,

$$\frac{\delta^{(r)} S_{gf}}{\delta \vartheta^\ell(t)} = \left(\frac{\delta^{(r)} S}{\delta \vartheta^\ell(t)}\right)_{gf} + \int dt' \left(\frac{\delta^{(r)} S}{\delta \vartheta^\star_k(t')}\right)_{gf} \frac{\delta^{(r)}}{\delta \vartheta^\ell(t)} \left(\frac{\delta^{(r)} \Psi_L}{\delta \vartheta^k(t')}\right) \ .$$

Hence

$$(S_{gf}, S)_{gf} = \frac{1}{2}(S, S)_{gf} + \int dt \int dt' \left(\frac{\delta^{(r)} S}{\delta \vartheta^\star_k(t')}\right)_{gf} K^{(r)}_{k\ell}(t', t) \left(\frac{\delta^{(l)} S}{\delta \vartheta^\star_\ell(t)}\right)_{gf} \ , \tag{12.70}$$

where

$$K^{(r)}_{k\ell}(t', t) = \frac{\delta^{(r)}}{\delta \vartheta^\ell(t)} \left(\frac{\delta \Psi_L}{\delta \vartheta^k(t')}\right) \ . \tag{12.71}$$

The second term on the rhs of (12.70) vanishes. The proof of this rests on the Grassmann signature of $\frac{\delta S}{\delta \vartheta^\star_k}$ and $\frac{\delta S}{\delta \vartheta^\star_\ell}$, as well as on the property

$$K^{(r)}_{\ell k}(t, t') = (-1)^{\epsilon_\ell \epsilon_k} K^{(r)}_{k\ell}(t', t) \ . \tag{12.72}$$

We thus finally arrive at the statement that

$$(S_{gf}, S)_{gf} = \frac{1}{2}(S, S)_{gf} \ . \tag{12.73}$$

Hence, if $(S,S)=0$, then S_{gf} is invariant under the transformations (12.68). The condition of BRST invariance of the gauge fixed action has thus been translated into the classical master equation for the field-antifield action.

Note that if $S[\vartheta,\vartheta^\star]$ is a linear functional of the anti-fields, then the BRST variations that leave the gauge fixed action invariant do not depend on the choice of gauge (i.e. on Ψ_L).

We complete the above analysis by demonstrating that the operator $\overset{\vee}{s}_\Psi$ defined in (12.68), and more generally in (12.69), is nilpotent *on shell*, i.e., that $\overset{\vee}{s}_\Psi\overset{\vee}{s}_\Psi f[\vartheta]=0$. To this effect consider a functional $f[\vartheta]$, depending only on the ϑ^ℓ's. From (12.69) it follows that

$$\overset{\vee}{s}_\Psi{}^2 f = ((f,S)_{gf},S)_{gf} \ .$$

The rhs is given by

$$((f,S)_{gf},S)_{gf} = \int dt \left(\frac{\delta^{(r)}f}{\delta\vartheta^\ell}\left(\frac{\delta^{(l)}S}{\delta\vartheta^\star_\ell}\right)_{gf}, S\right)_{gf} \ .$$

Making use of

$$(XY,S) = X(Y,S) + (-1)^{\epsilon_X\epsilon_Y}Y(X,S) \ ,$$

(see (12.5)), one verifies that

$$\begin{aligned} ((f,S)_{gf},S)_{gf} =& \int dt\ \left(\frac{\delta^{(r)}f}{\delta\vartheta^\ell(t)}\right)((\vartheta^\ell(t),S)_{gf},S)_{gf} \\ &+ \int dt\int dt'\ (-1)^{\epsilon_f\epsilon_{\vartheta^\star_\ell}}\left(\frac{\delta^{(l)}S}{\delta\vartheta^\star_\ell(t)}\right)_{gf} h_{\ell k}(t,t')\left(\frac{\delta^{(l)}S}{\delta\vartheta^\star_k(t')}\right)_{gf} \ , \end{aligned} \tag{12.74}$$

where

$$h_{\ell k}(t,t') = \frac{\delta^{(r)}}{\delta\vartheta^k(t')}\left(\frac{\delta^{(r)}f}{\delta\vartheta^\ell(t)}\right) \ ,$$

and

$$h_{\ell k}(t,t') = (-1)^{\epsilon_{\vartheta_k}\epsilon_{\vartheta_\ell}}h_{k\ell}(t',t) \ .$$

Making use of this symmetry one finds, after some algebra, that the second term on the rhs of (12.74) vanishes, so that

$$\overset{\vee}{s}{}^2_\Psi\, f \equiv ((f,S)_{gf},S)_{gf} = \int dt\ \left(\frac{\delta^{(r)}f}{\delta\vartheta^\ell(t)}\right)\overset{\vee}{s}{}^2_\Psi\,\vartheta^\ell(t) \ . \tag{12.75}$$

It therefore remains to be shown that $\overset{\vee}{s}{}^2_\Psi\vartheta^\ell=0$.

Consider the double bracket $((\vartheta^\ell, S)_{gf}, S)_{gf}$. We evidently have

$$\overset{\vee}{s}{}_\Psi^2 \vartheta^\ell = ((\vartheta^\ell, S)_{gf}, S)_{gf} = \left(\left(\frac{\delta^{(l)}S}{\delta\vartheta^\star_\ell}\right)_{gf}, S\right)_{gf}$$
$$= \int dt \left[\frac{\delta^{(r)}}{\delta\vartheta^k(t)}\left(\frac{\delta^{(l)}S}{\delta\vartheta^\star_\ell}\right)_{gf}\right]\left(\frac{\delta^{(l)}S}{\delta\vartheta^\star_k(t)}\right)_{gf} .$$

Noting that $\left(\frac{\delta^{(l)}S}{\delta\vartheta^\star_\ell}\right)_{gf}$ depends on ϑ also implicitly through the gauge fixing, this expression can also be written in the form

$$\overset{\vee}{s}{}_\Psi^2 \vartheta^\ell = \int dt \left[\left(\frac{\delta^{(r)}}{\delta\vartheta^k(t)}\left(\frac{\delta^{(l)}S}{\delta\vartheta^\star_\ell}\right)\right)\left(\frac{\delta^{(l)}S}{\delta\vartheta^\star_k(t)}\right)\right]_{gf} \tag{12.76}$$
$$+ \int dt \int dt' \left[\left(\frac{\delta^{(r)}}{\delta\vartheta^\star_{k'}(t')}\left(\frac{\delta^{(l)}S}{\delta\vartheta^\star_\ell}\right)\right)\frac{\delta^{(r)}}{\delta\vartheta^k(t)}\left(\frac{\delta\Psi_L}{\delta\vartheta^{k'}(t')}\right)\left(\frac{\delta^{(l)}S}{\delta\vartheta^\star_k(t)}\right)\right]_{gf} .$$

Next we rewrite the Ψ_L dependent term using the identity

$$\frac{\delta^{(r)}S_{gf}}{\delta\vartheta^{k'}} = \left(\frac{\delta^{(r)}S}{\delta\vartheta^{k'}}\right)_{gf} + \int dt \left(\frac{\delta^{(r)}S}{\delta\vartheta^\star_k(t)}\right)_{gf}\frac{\delta^{(r)}}{\delta\vartheta^{k'}}\left(\frac{\delta\Psi_L}{\delta\vartheta^k(t)}\right) . \tag{12.77}$$

After some rearrangements of factors and functional derivatives, we obtain

$$\int dt \left[\frac{\delta^{(r)}}{\delta\vartheta^k(t)}\left(\frac{\delta\Psi_L}{\delta\vartheta^{k'}}\right)\right]\left(\frac{\delta^{(r)}S}{\delta\vartheta^\star_k(t)}\right)_{gf} = (-1)^{\epsilon_{\vartheta_{k'}}}\left[\frac{\delta^{(r)}S_{gf}}{\delta\vartheta^{k'}} - \left(\frac{\delta^{(r)}S}{\delta\vartheta^{k'}}\right)_{gf}\right] .$$

Inserting this expression into (12.76) one finds after some further manipulations that,

$$\overset{\vee}{s}{}_\Psi^{\,2}\,\vartheta^\ell = (\vartheta^\ell, (S,S))_{gf} + \sum_k (-1)^{\epsilon_{\vartheta_k}} \int dt \left[\left(\frac{\delta^{(r)}}{\delta\vartheta^\star_k(t)}\left(\frac{\delta^{(l)}S}{\delta\vartheta^\star_\ell}\right)\right)\left(\frac{\delta^{(r)}S_{gf}}{\delta\vartheta^k(t)}\right)\right]_{gf} . \tag{12.78}$$

Hence, if S is a solution to the master equation, then $\overset{\vee}{s}{}_\Psi^2\vartheta^\ell = 0$ on the *mass shell*, i.e. when $\frac{\delta S_{gf}}{\delta\vartheta^\ell} = 0$. From (12.75) it follows that the same applies to $\overset{\vee}{s}{}_\Psi^{\,2} f[\vartheta]$.

The main task in constructing a gauge fixed BRST invariant action thus consists in obtaining a solution to the classical master equation. In practice this solution is generated iteratively by making the ansatz

$$S[\vartheta, \vartheta^\star] = S_{cl}[\vartheta] + \Delta_L[\vartheta, \vartheta^\star]$$

and expanding Δ_L in powers of the ϑ^ℓ's and $\vartheta^\star_\ell$'s, subject to the condition that it has even Grassmann parity and vanishing ghost number. Let us examine under what conditions the solution is of the above form with

$$\Delta_L[\vartheta, \vartheta^\star] = \int dt\ \vartheta^\star_\ell(\vartheta^\ell, S)\ .$$

We have

$$(S, S) = 2 \int dt\ \left[\frac{\delta^{(r)} S_{cl}}{\delta\vartheta^\ell(t)} \overset{\vee}{s}\vartheta^\ell(t) + \frac{\delta^{(r)}\Delta_L}{\delta\vartheta^\ell(t)} \overset{\vee}{s}\vartheta^\ell(t) \right]$$

with $\overset{\vee}{s}\vartheta^\ell$ the pullback of $\hat{s}\vartheta^\ell$ in phase space. The first term on the rhs vanishes since S_{cl} is BRST invariant. The second term can be written as $((\vartheta^\ell, S), S)$ *provided* S is linear in the antifields, in which case it reduces to $\frac{1}{2}((\vartheta, (S, S)))$ after use of Jacob's identity. Hence $(S, S) = 0$ is a solution in this case. Notice however that contrary to the case of phase-space discussed before, the condition of "linearity" is not guaranteed.

Example: The Yang-Mills theory

The $SU(3)$ Yang-Mills theory is an example satisfying all the assumptions *i) - iii)* made in the previous section. Hence we can immediately translate the partition function (12.51) to the case of the Yang-Mills theory:

$$\begin{aligned} Z^L_{YM} = \int DADBDcD\bar{c} \prod_{i,a,x} \delta\left(A^{a\star}_i(x) - \frac{\delta\Psi_L}{\delta A^a_i(x)} \right) \prod_{a,x} \delta\left(c^\star_a(x) - \frac{\delta\Psi_L}{\delta c^a(x)} \right) \\ \times \prod_{a,x} \delta\left(\bar{c}^{\star a}(x) - \frac{\delta\Psi_L}{\delta\bar{c}_a(x)} \right) e^{iS[A,c,\bar{c};A^\star,c^\star,\bar{c}^\star;B]}\ , \end{aligned}$$

where

$$S = \int d^4x\ \left(-\frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu} + A^{a\star}_\mu \overset{\vee}{s} A^\mu_a + c^\star_a \overset{\vee}{s} c^a + \bar{c}^{\star a} B_a \right)\ . \tag{12.79}$$

Notice that there is no $\bar{c}$-term, which, as we have already pointed out, only makes its appearance through the choice of fermion gauge fixing function. The variations $\overset{\vee}{s} A^a_\mu$ are determined by the symmetry of the classical action,

$$\overset{\vee}{s} A^\mu_a = \mathcal{D}^\mu_{ab} c^b\ , \tag{12.80}$$

where D^μ_{ab} is the covariant derivative (11.78). This is also evident from our analysis in the previous section. Less evident is the variation $\overset{\vee}{s} c^a$. Its variation is the analog in (12.43) with $f^{\alpha}_{\beta\gamma}$ replaced by the structure constants of $SU(3)$. Thus [18]

$$\overset{\vee}{s}c^a = -\frac{g}{2}f_{abc}c^b c^c \ , \tag{12.81}$$

and, as always,

$$\overset{\vee}{s}B_a = 0 \ . \tag{12.82}$$

Inserting these expressions in (12.79) one is led to

$$S = \int d^4x \left(-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + A^{\star a}_{\mu}\mathcal{D}^{\mu}_{ab}c^b - \frac{g}{2}f_{abc}c^{\star}_a c^b c^c + \bar{c}^{\star a}B_a\right) . \tag{12.83}$$

One explicitly checks that this action satisfies the master equation.

Note that the non-abelian structure of the YM-theory has induced a term bilinear in the ghost fields, carrying vanishing Grassmann parity and ghost number. This term will not contribute in gauges where Ψ_L (carrying ghost number -1) depends linearly on the Grassmann valued variables, and therefore linearly on the antighosts $\bar{c}^a$.

Next consider the gauge fixed action. The Lorentz gauge $\partial_\mu A^\mu_a = 0$ is evidently implemented by choosing for our Lagrangian "gauge fixing fermion" the following expression, linear in $\bar{c}_a$,

$$\Psi_L = \int d^4x \ \bar{c}_a(x)\partial^\mu A^a_\mu(x) \ .$$

The star-variables are then fixed as functions of A^μ_a, c^a and $\bar{c}_a$:

$$\begin{aligned}
&A^{\star\mu}_a(x) = -\partial^\mu \bar{c}_a(x) \ , \\
&c^{\star}_a(x) = 0 \ , \\
&\bar{c}^{\star a}(x) = \partial^\mu A^a_\mu(x) \ .
\end{aligned}$$

At this stage the antighosts have made their appearance, while the ghosts c^a are already present before having fixed the gauge. One is then led to the familiar Faddeev-Popov result (11.79) for the partition function in the Lorentz gauge.

The so-called "α-gauges" of 't Hooft are realized by the following choice of Ψ_L:

$$\Psi_L = \int d^4x \ \bar{c}_a(x)\left(\partial^\mu A^a_\mu(x) - \frac{\alpha}{2}B^a(x)\right) \ .$$

The gauge fixed YM-action then takes the form:

$$S_\alpha = \int d^4x \ \left(-\frac{1}{4}F^{\mu\nu}_a F^a_{\mu\nu} - \partial_\mu \bar{c}_a \mathcal{D}^{\mu}_{ab}c^b - (\frac{\alpha}{2}B^a - \partial^\mu A^a_\mu)B^a\right) \ .$$

[18] Recall from our analysis in the previous section that variations of configuration space variables that do not depend on the momenta π^a_i, P^a and $\bar{P}_a$ translate directly into $\overset{\vee}{s}$-variations in the field-antifield action.

This action is again invariant under the above mentioned transformations. But by integrating out the "Nakanishi-Lautrup field" B^a, the gauge fixed partition function becomes

$$Z_\alpha = \int DAD\bar{c}Dc\; e^{i\tilde{S}_\alpha} \ ,$$

where

$$\tilde{S}_\alpha = \int d^4x \ \left(-\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a - (\partial_\mu \bar{c}_a)\mathcal{D}^\mu_{ab}c^b + \frac{1}{2\alpha}(\partial^\mu A_\mu^a)^2\right) \ .$$

This action is invariant under the BRST variations

$$\begin{aligned} \overset{\vee}{s}A_a^\mu &= \mathcal{D}^\mu_{ab}c^b \ , \\ \overset{\vee}{s}c^a &= -\frac{g}{2}f_{abc}c^bc^c \ , \\ \overset{\vee}{s}\bar{c}_a &= -\frac{1}{\alpha}(\partial^\mu A_\mu^a) \ , \end{aligned}$$

which exhibit explicitly a gauge dependence through the parameter α.

12.5 The quantum master equation

In chapter 11 we have shown that the phase-space partition function of Fradkin and Vilkovisky is independent of the fermion gauge fixing function. This partition function was our starting point in section 3 for making the transition to configuration space. Hence the configuration space partition function (12.51) we arrived at should formally also exhibit this independence. It must however be kept in mind that this partition function was derived under a number of restrictive conditions, and in particular for a restricted class of fermion gauge fixing functions. As we now show, taking the independence of the partition function on the fermion gauge fixing function as our fundamental principle, we arrive in general at a modified (quantum) master equation.

For the sake of clarity we shall compactify our notation in the remaining part of this chapter. Any index labeling a field will be understood to include also the time variable. Thus $\varphi_\ell(t)$ will be simply denoted by φ_ℓ, and

$$\sum_\ell \int dt \to \sum_\ell \ ; \quad \sum_{\ell,\ell'} \int dt \int dt' \to \sum_{\ell,\ell'} \ .$$

Let us write the partition function (12.51) in the compact form (we drop from here on the superscript L on Z_ψ^L)

$$Z_\psi = \int D\vartheta D\vartheta^\star \prod_\ell \delta\left(\vartheta_\ell^\star - \frac{\delta\Psi_L}{\delta\vartheta^\ell}\right) e^{\frac{i}{\hbar}W[\vartheta,\vartheta^\star]} = \int \mathcal{D}\vartheta e^{\frac{i}{\hbar}W_\psi[\vartheta]} \ , \qquad (12.84)$$

where the gauge fixed action is given by

$$W_\psi[\vartheta] = W\left[\vartheta, \frac{\delta\Psi_L}{\delta\vartheta}\right] \ . \tag{12.85}$$

We have replaced S by W in anticipation of new results, and have introduced Planck's constant in order to keep track of quantum effects. Consider an infinitesimal change $\Psi_L \to \Psi_L + \Delta\Psi_L$. Recalling (12.4) we then have

$$Z_{\psi+\Delta\psi} - Z_\psi \approx \frac{i}{\hbar}\sum_\ell \int D\vartheta \left(\frac{\delta^{(r)}\Delta\Psi_L}{\delta\vartheta^\ell}\right)\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right)_{gf} e^{\frac{i}{\hbar}W_\psi} \ ,$$

or

$$\Delta_\psi Z_\psi \equiv Z_{\psi+\Delta\psi} - Z_\psi \approx \frac{i}{\hbar}\int D\vartheta \ e^{\frac{i}{\hbar}W_\psi}(\Delta\Psi_L, W)_{gf} \ . \tag{12.86}$$

Making use of [19]

$$\frac{\delta^{(r)}}{\delta\vartheta^\ell}(fg) = f\frac{\delta^{(r)}g}{\delta\vartheta_\ell} + (-1)^{\epsilon_{\vartheta_\ell}\epsilon_g}\frac{\delta^{(r)}f}{\delta\vartheta_\ell}g \ , \tag{12.87}$$

and of the fact that $\Delta\Psi_L$ has Grassmann parity $\epsilon = 1$, we obtain for the change in the partition function, after a partial integration,

$$\Delta_\psi Z_\psi = -\frac{i}{\hbar}\sum_\ell \int D\vartheta\Delta\Psi_L\left[\frac{i}{\hbar}\left(\frac{\delta^{(r)}W_\psi}{\delta\vartheta^\ell}\right)\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right)_{gf} + \frac{\delta^{(r)}}{\delta\vartheta^\ell}\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right)_{gf}\right]e^{\frac{i}{\hbar}W_\psi}$$

where we have made use of the fact that W_ψ does not depend on the antifields. Defining the action of $\tilde{\Delta}$ on W by

$$\tilde{\Delta}W = \sum_\ell \frac{\delta^{(r)}}{\delta\vartheta^\ell}\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right)_{gf} \ , \tag{12.88}$$

[19] Important relations between partial derivatives used here and in the following are:

$$\frac{\delta^{(l)}}{\delta\vartheta^\ell}(fg) = \frac{\delta^{(l)}f}{\delta\vartheta^\ell}g + (-1)^{\epsilon_f\epsilon_{\theta_\ell}}f\frac{\delta^{(l)}g}{\delta\vartheta^\ell}$$

$$\frac{\delta^{(r)}}{\delta\vartheta^\ell}(fg) = f\frac{\delta^{(r)}g}{\delta\vartheta^\ell} + (-1)^{\epsilon_g\epsilon_{\theta_\ell}}\frac{\delta^{(r)}f}{\delta\vartheta^\ell}g$$

$$\frac{\delta^{(l)}}{\delta\vartheta^\ell}F(f(\vartheta)) = \frac{\delta^{(l)}f}{\delta\vartheta^\ell}\frac{\delta^{(l)}F}{\delta f}$$

$$\frac{\delta^{(r)}}{\delta\vartheta^\ell}F(f(\vartheta)) = \frac{\delta^{(r)}F}{\delta f}\frac{\delta^{(r)}f}{\delta\vartheta^\ell}$$

$$\frac{\delta^{(l)}}{\delta\vartheta^\ell}\left(\frac{\delta^{(l)}\Delta\Psi}{\delta\vartheta^k}\right) = (-1)^{\epsilon_\ell\epsilon_k}\frac{\partial^{(r)}}{\partial\vartheta^\ell}\left(\frac{\delta^{(l)}\Delta\Psi}{\delta\vartheta^k}\right) \ ,$$

where in the last equation we have made use of the fact that $\epsilon_{\Delta\Psi} = -1$.

we conclude that if Z_ψ is not to depend on Ψ (gauge independence), then $W(\vartheta,\vartheta^\star)$ must be a solution to

$$i\hbar\tilde{\Delta}W = (W_\psi, W)_{gf} \ . \tag{12.89}$$

This is not yet the "quantum master equation" in the form given in the literature. In fact the lhs of (12.89), as well as the rhs can be further simplified. Consider first the lhs:

$$\frac{\delta^{(r)}}{\delta\vartheta^\ell}\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right)_{gf} = \left[\frac{\delta^{(r)}}{\delta\vartheta^\ell}\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right) + \sum_{\ell'}\frac{\delta^{(r)}}{\delta\vartheta^\star_{\ell'}}\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right)K^{(r)}_{\ell'\ell}\right]_{gf} \ ,$$

where the matrix $K^{(r)}_{\ell'\ell}$ has been defined in (12.71), with the symmetry property (12.72). Hence

$$\tilde{\Delta}W = \sum_\ell(-1)^{\epsilon_{\vartheta_\ell}+1}\frac{\delta^{(r)}}{\delta\vartheta^\ell}\left(\frac{\delta^{(r)}}{\delta\vartheta^\star_\ell}W\right)_{gf} = [\Delta_{op}W + \sum_{\ell,\ell'}A^{(r)}_{\ell'\ell}K^{(r)}_{\ell'\ell}]_{gf} \tag{12.90}$$

where

$$\Delta_{op} \equiv \sum_\ell(-1)^{\epsilon_{\vartheta_\ell}+1}\frac{\delta^{(r)}}{\delta\vartheta^\ell}\frac{\delta^{(r)}}{\delta\vartheta^\star_\ell} \tag{12.91}$$

and

$$A^{(r)}_{\ell'\ell} = (-1)^{\epsilon_{\vartheta_\ell}+1}\frac{\delta^{(r)}}{\delta\vartheta^\star_{\ell'}}\left(\frac{\delta^{(r)}}{\delta\vartheta^\star_\ell}W\right) \ ,$$

with the symmetry property

$$A^{(r)}_{\ell'\ell} = -(-1)^{\epsilon_{\vartheta_\ell}\epsilon_{\vartheta_{\ell'}}}A^{(r)}_{\ell\ell'} \ . \tag{12.92}$$

Because of the symmetry properties (12.72) and (12.92), the sum on the rhs of (12.90) vanishes, so that

$$\tilde{\Delta}W = (\Delta_{op}W)_{gf}. \tag{12.93}$$

Finally let us rewrite the rhs of (12.89). It can be written in the form

$$\begin{aligned}(W_\psi, W)_{gf} &= \sum_\ell\left(\frac{\delta^{(r)}W_\psi}{\delta\vartheta^\ell}\right)\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right)_{gf} \\ &= \sum_\ell\left[\left(\frac{\delta^{(r)}W}{\delta\vartheta^\ell}\right)_{gf} + \sum_{\ell'}\left(\frac{\delta^{(r)}W}{\delta\vartheta^\star_{\ell'}}\right)_{gf}\frac{\partial^{(r)}}{\partial\vartheta^\ell}\left(\frac{\delta\Psi_L}{\delta\vartheta^{\ell'}}\right)\right]\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right)_{gf} \ ,\end{aligned}$$

or after taking proper account of the phases,

$$(W_\psi, W)_{gf} = \frac{1}{2}\sum_\ell \left[\left(\frac{\delta^{(r)}W}{\delta\vartheta^\ell}\right)\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right) - \left(\frac{\delta^{(r)}W}{\delta\vartheta^\star_\ell}\right)\left(\frac{\delta^{(l)}W}{\delta\vartheta^\ell}\right)\right]_{gf}$$
$$+ \sum_{\ell\ell'}\left[(-1)^{\epsilon_{\vartheta_\ell}+1}\left(\frac{\delta^{(r)}W}{\delta\vartheta^\star_{\ell'}}\right)K^{(r)}_{\ell'\ell}\left(\frac{\delta^{(r)}W}{\delta\vartheta^\star_\ell}\right)\right]_{gf},$$

that is

$$(W_\psi, W)_{gf} = \frac{1}{2}(W,W)_{gf} + \frac{1}{2}\sum_{\ell\ell'}(-1)^{\epsilon_{\vartheta^\star_\ell}\epsilon_{\vartheta^\star_{\ell'}}}\left[K^{(r)}_{\ell'\ell}B^{(r)}_{\ell'\ell} + K^{(r)}_{\ell\ell'}B^{(r)}_{\ell\ell'}\right] \quad (12.94)$$

where

$$B_{\ell'\ell} = (-1)^{\epsilon_{\vartheta_\ell}+1}\left(\frac{\delta^{(r)}W}{\delta\vartheta^\star_{\ell'}}\right)_{gf}\left(\frac{\delta^{(r)}W}{\delta\vartheta^\star_\ell}\right)_{gf}.$$

Making use of the symmetry property (12.72), and of

$$B^{(r)}_{\ell\ell'} = -(-1)^{\epsilon_{\vartheta_\ell}\epsilon_{\vartheta_{\ell'}}}B^{(\ell)}_{\ell'\ell},$$

one then finds that the sum on the rhs of (12.94) vanishes. Hence, analogous to (12.73), we have

$$(W_\psi, W)_{gf} = \frac{1}{2}(W,W)_{gf}. \quad (12.95)$$

Since the gauge fixing function is arbitrary, this leaves us with the quantum master equation written in form:

$$i\hbar\Delta_{op}W = \frac{1}{2}(W,W), \quad (12.96)$$

where Δ_{op} has been defined in (12.91).

Equation (12.96) must hold if we are dealing with a true gauge theory on quantum level. It is frequently written also in the equivalent form

$$\Delta_{op}e^{\frac{i}{\hbar}W} = 0.$$

12.5.1 An alternative derivation of the quantum master equation

It is instructive to derive the quantum master equation in an alternative way.

Consider the infinitesimal variable transformation

$$\vartheta^\ell \to \tilde\vartheta^\ell = \vartheta^\ell + \frac{1}{\hbar}(\vartheta^\ell, W)_{gf}\Delta\Psi_L, \quad (12.97)$$

where $\Delta\Psi_L$ is in general a *global Grassmann* valued functional of the fields, independent of (space) time. If $\Delta\Psi_L$ did not depend on the fields, and if $(W,W) = 0$, then (12.97) would be a BRST transformation on Lagrangian level. [20] Consider the identity

$$
\begin{aligned}
Z_\psi &= \int D\tilde{\vartheta}\, e^{\frac{i}{\hbar}W_\psi[\tilde{\vartheta}]} \\
&= \int D\vartheta\, J[\vartheta]\, e^{\frac{i}{\hbar}W_\psi[\vartheta + \frac{1}{\hbar}(\vartheta, W)_{gf}\Delta\Psi_L]}\,,
\end{aligned} \tag{12.98}
$$

where $J[\vartheta]$ is the Jacobian of the transformation (12.97). To calculate the Jacobian we must compute the determinant of the matrix [21]

$$
\frac{\partial^{(l)}\tilde{\vartheta}^\ell}{\partial\vartheta^k} = \delta^\ell_k + \frac{1}{\hbar}A^\ell_k
$$

with

$$
A^\ell_k = \frac{\delta^{(l)}}{\delta\vartheta^k}\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right)_{gf}\Delta\Psi_L + (-1)^{\epsilon_{\vartheta_k}\epsilon_{\vartheta^\star_\ell}}\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right)_{gf}\frac{\delta\Delta\Psi_L}{\delta\vartheta^k}\,,
$$

where we have made use of (12.4). For A^ℓ_k infinitesimal, the determinant is given by

$$
det(1+A) \approx 1 + TrA\ .
$$

Taking into account that for bosons (fermions) the Jacobian is given by the determinant (inverse determinant) of $(1+A)$, [22] we conclude that

$$
\begin{aligned}
J[\vartheta] &\approx 1 + \frac{1}{\hbar}\sum_\ell (-1)^{\epsilon_{\vartheta_\ell}}\left[\frac{\delta^{(l)}}{\delta\vartheta^\ell}\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right)_{gf}\Delta\Psi_L + \left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right)_{gf}\frac{\delta\Delta\Psi_L}{\delta\vartheta^\ell}\right] \\
&\approx 1 + \frac{1}{\hbar}\sum_\ell\left[\frac{\delta^{(r)}}{\delta\vartheta^\ell}\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right)_{gf}\Delta\Psi_L - \frac{\delta\Delta\Psi_L}{\delta\vartheta^\ell}\left(\frac{\delta^{(l)}W}{\delta\vartheta^\star_\ell}\right)_{gf}\right]
\end{aligned}
$$

[20] The following argumentation does not depend on whether the quantum theory is anomalous or not.

[21] Recall that the "time" has been absorbed into the "discrete" indices labeling the fields.

[22] A simple argument demonstrates this. Consider in particular the case of one variable:

$$
v' = bv\,.
$$

Hence

$$
\int dv'\, v' = \int J dv (bv)\,.
$$

Using the Grassmann rules for integration, we conclude that $J = 1/b$. Note that TrA includes the integration over time.

or

$$J[\vartheta] \approx \left(1 + \frac{1}{\hbar}\left[(\tilde{\Delta}W)\Delta\Psi_L - (\Delta\Psi_L, W)_{gf}\right]\right) , \qquad (12.99)$$

where we have used the Grassmann nature of $\Delta\Psi_L$, as well as (12.4) in order to eliminate the phase $(-1)^{\epsilon_\varepsilon}$.

Note that for $\Delta\Psi_L \to \epsilon$, $(\Delta\Psi_L, W) = 0$, and $J[\theta] \approx 1 + \frac{1}{\hbar}(\tilde{\Delta}W)\epsilon$. Hence a non-invariance of the functional measure under BRST transformations is tied to a non-vanishing $\tilde{\Delta}W$, or $(\Delta_{op}W)_{gf}$.

Inserting the Jacobian (12.99) into the rhs of (12.98), one is led to first order in $\Delta\Psi_L$ to the identity

$$\int D\vartheta e^{iW_\psi[\vartheta]}\frac{1}{\hbar}\left[\Big(\tilde{\Delta}_{op}W - (W_\psi, W)\Big)_{gf}\Delta\Psi_L - (\Delta\Psi_L, W)_{gf}\right] \approx 0 \,.$$

Accordng to (12.86) the last contribution in this expression is nothing but $i\Delta_\psi Z_\psi$. We therefore conclude that if $\Delta_\psi Z_\psi = 0$, i.e., if Z_ψ is not to depend on the gauge fixing function, then equation (12.89) must hold. The remaining steps leading to (12.96) are just as described before.

From (12.99) we also notice, that if $(\Delta\Psi_L, W)_{gf} = i\Delta_\psi Z_\psi = 0$, then a violation of the classical master equation is connected with a non-invariance of the integration measure under transformations of the form (12.97).

The presence of $\hbar$ in the quantum master equation signalizes that the violation of the classical master equation is of quantum nature, and suggests a solution in the form of a power series in $\hbar$. One is thus led to the Ansatz

$$W = S[\vartheta, \vartheta^\star] + \sum_p \hbar^p M_p[\vartheta, \vartheta^\star] \,. \qquad (12.100)$$

Substituting this expression into the quantum master equation (12.96), and comparing powers in $\hbar$, one arrives at an (infinite) set of coupled equations:

$$\begin{aligned}
&(S, S) = 0 \,, \\
&(M_1, S) = i\Delta_{op}S \,, \\
&(M_p, S) = i\Delta_{op}M_{p-1} - \frac{1}{2}\Sigma_{q=1}^{p-1}(M_q, M_{p-q}) \,, \quad p \geq 2 \,.
\end{aligned} \qquad (12.101)$$

The first equation states that the "classical" contribution $S[\vartheta, \vartheta^\star]$ is classically BRST invariant, with S the generator of BRST transformations.

Batalin and Vilkovisky have shown that the classical master equation always has a solution. If on the other hand one cannot find a *local* solution to the coupled set of equations above, this signalizes the existence of a gauge anomaly [Troost 1990]. We shall say more about this in section 6.

12.5.2 Gauge invariant correlation functions

Let us now turn to the condition which correlation functions must satisfy in order to be independent of the choice of the gauge fixing fermion Ψ_L. Within the field-antifield formalism correlation functions are computed from the following integral

$$\begin{aligned} < X >_\psi &= \int D\vartheta D\vartheta^\star \prod_\ell \delta\left(\vartheta^\star_\ell - \frac{\delta\Psi_L}{\delta\vartheta^\ell}\right) X[\vartheta,\vartheta^\star] e^{\frac{i}{\hbar}W[\vartheta,\vartheta^\star]} \\ &= \int D\vartheta X_\psi[\vartheta] e^{\frac{i}{\hbar}W_\psi[\vartheta]} \ , \end{aligned} \tag{12.102}$$

where

$$X_\psi[\vartheta] = X[\vartheta, \frac{\delta\Psi_L}{\delta\vartheta}] \ .$$

The change in $< X >_\psi$ induced by $\Psi_L \to \Psi_L + \Delta\Psi_L$ is given by

$$< X >_{\psi+\Delta\psi} - < X >_\psi = \int D\vartheta \left[\Delta_\psi X_\psi + \frac{i}{\hbar} X_\psi \Delta_\psi W_\psi\right] e^{\frac{i}{\hbar}W_\psi}$$

where, generically, $F_\psi[\theta] = F[\theta, \frac{\delta\Psi_L}{\partial\theta}]$, and

$$\begin{aligned} \Delta_\psi F_\psi &= F[\vartheta, \frac{\delta(\Psi_L + \Delta\Psi_L)}{\delta\vartheta}] - F[\vartheta, \frac{\delta\Psi_L}{\partial\vartheta}] \\ &= \left(\frac{\delta^{(r)}F}{\delta\vartheta^\star_\ell}\right)_{gf} \left(\frac{\delta\Delta\Psi_L}{\delta\vartheta^\ell}\right) \end{aligned}$$

Note that we have expressed the changes $\Delta_\psi X_\psi$ and $\Delta_\psi W_\psi$ in terms of right derivatives. This turns out to be convenient. After performing a partial integration, making use of (12.87), one obtains

$$\begin{aligned} < X >_{\psi+\Delta\psi} - < X >_\psi &= \int D\vartheta (-1)^{\epsilon_{\vartheta_\ell}+1} \frac{\delta^{(r)}}{\delta\vartheta^\ell}\left[\left(\frac{\delta^{(r)}X}{\delta\vartheta^\star_\ell}\right)_{gf} e^{\frac{i}{\hbar}W_\psi}\right]\Delta\Psi_L \\ &+ \frac{i}{\hbar}\int D\vartheta (-1)^{\epsilon_{\vartheta_\ell}+1} \frac{\delta^{(r)}}{\delta\vartheta^\ell}\left[X_\psi \left(\frac{\delta^{(r)}W}{\delta\vartheta^\star_\ell}\right)_{gf} e^{\frac{i}{\hbar}W_\psi}\right]\Delta\Psi_L \ . \end{aligned}$$

In carrying out the following partial differentiations, the reader should remember, when applying the Grassmann differentiation rules, that all the above derivatives operate from the right. One then readily finds that

$$\begin{aligned} < X >_{\psi+\Delta\psi} - < X >_\psi &= \int D\vartheta\ \Delta\Psi_L \left[X_\psi\left(\tilde\Delta W + \frac{i}{\hbar}(W_\psi, W)_{gf}\right)\right] e^{\frac{i}{\hbar}W_\psi} \\ &+ \int D\vartheta\ \Delta\Psi_L \left[\tilde\Delta X + \frac{i}{\hbar}\left((X, W_\psi) + (X_\psi, W)\right)_{gf}\right] e^{\frac{i}{\hbar}W_\psi} \ , \end{aligned} \tag{12.103}$$

where $\tilde{\Delta}$ is defined in (12.88). Hence the ψ-independence of a correlator X requires that

$$\tilde{\Delta} W + \frac{i}{\hbar}(W_\psi, W)_{gf} = 0 \ ,$$

$$\tilde{\Delta} X + \frac{i}{\hbar}[(X, W_\psi) + (X_\psi, W)]_{gf} = 0 \ . \tag{12.104}$$

These expressions can be further simplified. Consider the rhs of (12.104).

$$(X_\psi, W)_{gf} = \sum_\ell \left[\left(\frac{\delta^{(r)} X}{\delta \vartheta^\ell} \right) + \sum_k \left(\frac{\delta^{(r)} X}{\delta \vartheta^\star_k} \right) \frac{\delta^{(r)}}{\delta \vartheta^\ell} \left(\frac{\delta \Psi_L}{\delta \vartheta^k} \right) \right]_{gf} \left(\frac{\delta^{(\ell)} W}{\delta \vartheta^\star_\ell} \right)_{gf}$$

and

$$(X, W_\psi)_{gf} = -\sum_\ell \left(\frac{\delta^{(r)} X}{\delta \vartheta^\star_\ell} \right)_{gf} \left[\left(\frac{\delta^{(l)} W}{\delta \vartheta^\ell} \right) + \sum_k \frac{\delta^{(l)}}{\delta \vartheta^\ell} \left(\frac{\delta \Psi_L}{\delta \vartheta^k} \right) \left(\frac{\delta^{(l)} W}{\delta \vartheta^\star_k} \right) \right]_{gf} .$$

Hence

$$[(X_\psi, W) + (X, W_\psi)]_{gf} = \left[(X, W)_{gf} + \sum_{\ell, k} \left(\frac{\delta^{(r)} X}{\delta \vartheta^\star_k} \right) [K^{(r)}_{k\ell} - K^{(l)}_{k\ell}] \left(\frac{\delta^{(l)} W}{\delta \vartheta^\star_\ell} \right) \right]_{gf}$$

where $K^{(r)}_{k\ell}$ has been defined in (12.71) and

$$K^{(l)}_{k\ell} = \frac{\delta^{(l)}}{\delta \vartheta^k} \left(\frac{\delta \Psi_L}{\delta \vartheta^\ell} \right) .$$

But $K^{(r)}_{k\ell} = K^{(l)}_{k\ell}$. Hence

$$[(X_\psi, W) + (X, W_\psi)]_{gf} = (X, W)_{gf} \ .$$

This also holds for X replaced by any other function F. Condition (12.104) on $X(\vartheta, \vartheta^\star)$ thus reduces to

$$\tilde{\Delta} X + \frac{i}{\hbar}(X, W)_{gf} = 0 \, .$$

Now, we have seen that $\tilde{\Delta} F = (\Delta_{op} F)_{gf}$ for arbitrary $F(\vartheta, \vartheta^\star)$. Furthermore, the Ψ_L of the "gf" surface $\vartheta^\star_\ell - \frac{\delta \Psi_L}{\delta \vartheta^\ell} = 0$ is completely arbitrary. Hence we conclude that the quantum field-antifield action W, and the "observables" $X(\vartheta, \vartheta^\star)$ must be solutions to

$$\Delta_{op} W + \frac{i}{2\hbar}(W, W) = 0 \ , \quad \Delta_{op} X + \frac{i}{\hbar}(X, W) = 0 \ . \tag{12.105}$$

These are the conditions that X in (12.102) and W must satisfy in order to yield an expectation value which does not depend on the gauge choice.

A bonafide (gauge invariant) operator X satisfying (12.105) is not trivial to construct. Similarly as in the case of W, let us expand X in a power series of $\hbar$:

$$X = X_0 + \sum_{q=1}^{\infty} \hbar^q X_q \,. \tag{12.106}$$

Inserting this expression, as well as (12.100), into (12.105), one is led to the following iterative sheme:

$$\begin{aligned} (X_0, S) &= 0 \\ (X_0, M_1) + (X_1, S) &= i\Delta_{op} X_0 \\ \sum_{q=1}^{p-1} (X_q, M_{p-q}) + (X_0, M_p) + (X_p, S) &= i\Delta_{op} X_{p-1} \quad ; \; p \geq 2 \,. \end{aligned} \tag{12.107}$$

This concludes the very detailed proofs of results usually just stated in the literature.

The following example is meant to provide some insight into the significance and role of the quantum master equation, and its solution.

12.6 Anomalous gauge theories. The chiral Schwinger model

The field-antifield formalism discussed above was applied to Lagrangians with a local symmetry, and whose quantum partition function also respects this symmetry. Anomalous gauge theories, where the gauge invariance is broken on quantum level, do a priori not fall into this class. If the theory can however be embedded on quantum level into a gauge theory, then the field-antifield formalism can again be applied to this embedded theory. If there exists a gauge in which the partition function of the embedded formulation reduces to that of the anomalous theory, then gauge invariant observables can be set into one to one correspondence with the observables of the anomalous theory.

In the following we do not embarque on a general discussion of anomalous gauge theories within the BV framework, since it falls outside the scope of this book. For an elaborate discussion of this subject we refer the reader to [Troost 1990], [De Jonghe 1993] and [Gomis 1993/94/95]. Here we merely use the chiral Schwinger model (CSM) as a toy model in order to get some insight into the problem. The starting point for our analysis will be a gauge invariant embedded version thereof. According to our discussion carried out in the previous sections, we should find that, irrespective of the embedding procedure, the corresponding field-antifield action is a solution to the quantum master equation.

In the following we shall consider two embedded formulations of the chiral Schwinger model. In the approach to be discussed first, the partition function of the embedded theory is expressed in terms of the original fields and an auxiliary bosonic field $\theta(x)$. In this case, both, the functional integration measure and the action are not invariant under local transformations, but the corresponding partition function is. According to our discussion in section 12.5.1 the non-invariance of the functional integration measure should manifest itself in a non-vanishing contribution $\Delta_{op}W$ in (12.96). This contribution should be closely connected to the gauge anomaly. In the second approach the fermionic action is replaced by an equivalent bosonic action (bosonization). In this case the corresponding gauge invariant partition function is expressed entirely in terms of bosonic fields, and both, the measure and the embedded action are gauge invariant. We now present the details.

The chiral Schwinger model (CSM) corresponds to electrodynamics of massless fermions in 1+1 dimensions, in which only one of the chiralities of the fermions couples to the gauge field. It is defined by the classical action [23]

$$S_{ferm} = \int d^2x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}\gamma^\mu \left(i\partial_\mu + 2\sqrt{\pi} e A_\mu P_L \right) \psi \right] , \tag{12.108}$$

where

$$P_L = \frac{1}{2}(1 - \gamma_5)$$

is the projector on the left hand modes. This action is invariant under the (infinitesimal) chiral gauge transformation

$$\begin{aligned} \delta A^\mu(x) &= \frac{1}{e}\partial^\mu \alpha(x) \ , \\ \delta\psi(x) &= 2i\sqrt{\pi} P_L \psi(x) \alpha(x) \ , \\ \delta\bar{\psi}(x) &= -2i\sqrt{\pi}\bar{\psi}(x) P_R \alpha(x) \, , \end{aligned} \tag{12.109}$$

where $P_R = \frac{1}{2}(1 + \gamma_5)$. This is not the case for the partition function,

$$Z = \int DA^\mu \int D\bar{\psi} D\psi \, e^{iS_{ferm}[A,\bar{\psi},\psi]} \, , \tag{12.110}$$

[23] We set $\hbar = 1$. The factor $2\sqrt{\pi}$ multiplying the charge e has been introduced in order to conform to the conventions of chapter 15 of [Abdalla 2001]. For a comprehensive discussion of the chiral Schwinger model and the material of this section we refer the reader to this chapter.

since the fermion functional integration measure is not invariant under this transformation. In fact, under the chiral gauge transformation (12.109), the measure transforms as follows [Abdalla 2001],

$$D\bar{\psi}D\psi \to e^{i\alpha_1[A,\alpha]}D\bar{\psi}D\psi \ , \tag{12.111}$$

where $\alpha_1[A,\alpha]$ is the functional

$$\alpha_1[A,\alpha] = \int d^2x \left[\frac{1}{2}(a-1)\partial_\mu\alpha\partial^\mu\alpha + eA^\mu((a-1)\partial_\mu + \tilde{\partial}_\mu)\alpha\right] \ , \tag{12.112}$$

($\tilde{\partial}_\mu \equiv \epsilon_{\mu\nu}\partial^\nu$, $\epsilon_{01} = 1$) with the following 1-cocycle property,

$$\alpha_1[A + \frac{1}{e}\partial\beta, \alpha - \beta] = \alpha_1[A,\alpha] - \alpha_1[A,\beta] \ . \tag{12.113}$$

The parameter a in (12.112) is the Jackiw-Rajaraman parameter [Jackiw 1985], reflecting an ambiguity in defining the fermionic measure. This non-invariance of the fermionic measure becomes manifest upon performing the integration over the fermion fields in (12.110). One finds

$$Z = \int DA^\mu \, e^{iW^{(L)}[A]} e^{-i\int d^2x \, \frac{1}{4}F^{\mu\nu}F_{\mu\nu}} \ , \tag{12.114}$$

where $W^{(L)}[A] = \ln \det(i\partial\!\!\!/ + 2\sqrt{\pi}eA\!\!\!/P_L)$, and given by [24]

$$W^{(L)}[A] = \frac{ae^2}{2}\int d^2x \, A^\mu A_\mu - \frac{e^2}{2}\int\int A^\mu(\partial_\mu + \tilde{\partial}_\mu)\frac{1}{\Box}(\partial^\nu + \tilde{\partial}^\nu)A_\nu \, . \tag{12.115}$$

Note that this expression is non-local in the gauge fields and manifestly exhibits the breaking of gauge invariance on quantum level. In fact, the gauge transform of $W^{(L)}[A]$ and α_1 are related by

$$W^{(L)}[A + \frac{1}{e}\partial\alpha] - W^{(L)}[A] = \alpha_1[A,\alpha] \, .$$

Since the partition function (12.114) is not gauge invariant, it does not admit a BV representation. In order to apply the field-antifield formalism we shall thus have to turn the theory first into a first class theory, by embedding the system into an extended space involving an additional scalar field $\theta(x)$. The transformation law (12.111) for the fermionic measure, and the cocycle property (12.113), suggest the following form for the embedded action:

$$S^{emb}_{ferm}[A,\psi,\bar{\psi},\theta] = S_{ferm} + \alpha_1[A,\theta] \, . \tag{12.116}$$

[24] We use here a compact notation for the non-local second integral.

The corresponding partition function

$$Z^{emb} = \int DAD\theta D\bar{\psi}D\psi \; e^{i(S_{ferm}+\alpha_1[A,\theta])} \;, \tag{12.117}$$

is invariant under the variations (12.109), together with

$$\delta\theta = -\alpha \;. \tag{12.118}$$

Indeed, the non-invariance of the fermionic measure in (12.111) is compensated by the corresponding change in $\alpha_1[A,\theta]$, as seen from (12.113):

$$\delta\alpha_1[A,\theta] = -\alpha_1[A,\alpha] \;.$$

In the gauge $\theta = 0$, (12.117) reduces to the anomalous partition function (12.110) of interest.

The θ dependent term $\alpha_1[A,\theta]$ in (12.116) is generally referred to as the "Wess-Zumino action" for the abelian model in question. The introduction of additional gauge degrees of freedom to arrive at a gauge invariant partition function for anomalous gauge theories was first considered by Wess and Zumino [Wess 1971], and further explored by Faddeev and Shatashvili [Faddeev 1986], and by Harada and Tsutsui [Harada 1987]. Within the present context it was also introduced in [Gomis 1994, Braga 1993] from more general point of views, in order to obtain a local solution to the quantum master equation in the extended space.

While the action and functional integration measure $D\bar{\psi}D\psi$ both break the above gauge invariance, the partition function (12.117) does not. We are therefore allowed to use a gauge- fixing procedure á la BV. Correspondingly we write for the embedded gauge fixed field-antifield partition function,

$$Z^{emb}_{gf} = \int DBD\theta DA \int D\bar{\psi}D\psi \int D\bar{c}Dc \; e^{iW_\psi[A,\psi,\bar{\psi},\theta,c,\bar{c},\theta;B]} \,, \tag{12.119}$$

where

$$W_\psi = S^{emb}_{ferm} + \int d^2x \, [A^\star_\mu \overset{\vee}{s} A^\mu + \psi^\star \overset{\vee}{s} \psi + \bar{\psi}^\star \overset{\vee}{s} \bar{\psi} + \theta^\star \overset{\vee}{s} \theta + c^\star \overset{\vee}{s} c + B\bar{c}^\star]_{gf} \;.$$

As always we have included a term $B\bar{c}^\star$ for gauge fixing purposes.

The explicit form for the gauge fixed expression is obtained by choosing a "fermion gauge fixing function", Ψ_L", and gauge fixing the antifields according to

$$A^\star_\mu = \frac{\delta\Psi_L}{\delta A^\mu} \;, \quad \theta^\star = \frac{\delta\Psi_L}{\delta\theta} \;, \quad \bar{c}^\star = \frac{\delta\Psi_L}{\delta c}$$
$$\psi^\star_\alpha = \frac{\delta\Psi_L}{\delta\psi_\alpha} \;, \quad \bar{\psi}^\star_\alpha = \frac{\delta\Psi_L}{\delta\bar{\psi}_\alpha} \;, \quad c^\star = \frac{\delta\Psi_L}{\delta c} \;.$$

Since the classical Lagrangian describes a rank-one theory with an abelian gauge structure, we expect that $\overset{\vee}{s}c = 0$, and that the remaining BRST transformations are completely determined by the symmetry transformations (12.109) and (12.118). Hence

$$W_\psi = S^{emb}_{ferm} + \int d^2x\,[A^\star_\mu \frac{1}{e}\partial^\mu c + 2i\sqrt{\pi}\psi^\star P_L\psi c - 2i\sqrt{\pi}\bar{\psi}P_R\bar{\psi}^\star c - \theta^\star c + B\bar{c}^\star]_{gf}\ . \tag{12.120}$$

Note that in the gauge $\theta = 0$, corresponding to the choice $\Psi_L = \int d^2x\ \bar{c}\theta$ of the gauge fixing fermion, $A^\star, \psi^\star, \bar{\psi}^\star$ and $c^\star$ vanish, while $\theta^\star = \bar{c}$. Since $\int D\bar{c}Dc\ \exp\{-i\int d^2x\,\bar{c}(x)c(x)\}$ is just a constant, we thus recover in this gauge the original partition function (12.110).

12.6.1 Quantum Master equation and the anomaly

Consider now the antibracket (W,W), where W is the non-gauge fixed field-antifield action in the embedding space.

$$W = S^{emb}_{ferm} + \int d^2x\,[A^\star_\mu \frac{1}{e}\partial^\mu c + 2i\sqrt{\pi}\psi^\star P_L\psi c - 2i\sqrt{\pi}\bar{\psi}P_R\bar{\psi}^\star c - \theta^\star c + B\bar{c}^\star]\ .$$

For (W,W) one finds after a partial integration, that

$$\begin{aligned}\frac{1}{2}(W,W) = \int d^2x\Big[&2\sqrt{\pi}\bar{\psi}\gamma_\mu P_L\psi\partial^\mu c + ((a-1)\partial_\mu\theta + \tilde{\partial}_\mu\theta)\partial^\mu c\\ &- 2\sqrt{\pi}\bar{\psi}\gamma^\mu P_L\partial_\mu(\psi c) - 2\sqrt{\pi}\bar{\psi}\gamma^\mu P_L c\partial_\mu\psi\\ &+ 2ie\pi\bar{\psi}\gamma^\mu P_L A_\mu(c\psi) + 2ie\pi\bar{\psi}\gamma^\mu P_L A_\mu\psi c\\ &+ [(a-1)\Box\theta + e((a-1)\partial^\mu A_\mu + \tilde{\partial}_\mu A^\mu)]c\Big]\ .\end{aligned}$$

Noting that $\partial^\mu\tilde{\partial}_\mu = 0$, one obtains

$$\frac{1}{2}(W,W) = e\int d^2x\, c(x)[(a-1)\partial^\mu + \tilde{\partial}_\mu]A^\mu(x)\ .$$

Comparing this with (12.96) we are led to the identification (presently, $\hbar = 1$)

$$i\Delta_{op}W = \int d^2x\, c(x)\mathcal{A}(x)\ , \tag{12.121}$$

where $\mathcal{A}$ is the so-called *consistent* anomaly,

$$\mathcal{A}(x) = e[(a-1)\partial_\mu + \tilde{\partial}_\mu]A^\mu(x)\,, \tag{12.122}$$

which expresses the failure of the covariant conservation of the chiral current.

$$\partial_\mu J^\mu_{ch}(x) = \mathcal{A}(x)\,.$$

The computation of the singular lhs of (12.121) is actually ambiguous, and leads to various forms of the anomaly, depending on the regularization procedure [Braga 1991]. The consensus is however, that it should be given by the consistent anomaly. Hence, as expected, the field-antifield partition function of the embedded CSM satisfies the quantum master equation, although neither the embedded action (12.120), nor the measure are gauge invariant. This is consistent with our analysis in section 5, which only demanded the partition function not to depend on the choice of gauge.

As already pointed out, a non-vanishing $\Delta_{op}W$ is expected to be tied to a non-invariance of the integration measure under BRST transformations. Indeed, for a transformation of the form (12.97), with $\Delta\Psi_L$ a constant Grassmann valued infinitesimal parameter ϵ, the Jacobian in (12.99), written in exponential form, becomes (for $\hbar = 1$)

$$J \approx e^{\epsilon\Delta_{op}W}$$

or with (12.121),

$$J \approx e^{-i\epsilon\int d^2x c(x)\mathcal{A}(x)}\,.$$

Noting that for the Grassmann valued c we have $\partial_\mu c\partial^\mu c = 0$, we see that this Jacobian is in agreement with (12.111) upon setting $\alpha = \epsilon c$. Hence neither (W, W) nor $\Delta_{op}W$ vanish in the present case, while the quantum master equation is satisfied. Thus, in this embedded formulation, the anomaly of the CSM manifests itself through a non-vanishing $\Delta_{op}W$, i.e. a non-trivial Jacobian.

For the case where the gauge fixing fermion functional Ψ_L is independent of θ, we can perform the integration over θ in (12.119), and are led to the gauge fixed partition function

$$\begin{aligned} Z = \int DADB \int D\psi D\bar{\psi} DcDc\; e^{i(S_{ferm}+M_1)} \\ \times\, e^{\int d^2x\; [A^\star_\mu \frac{1}{e}\partial^\mu c + 2i\sqrt{\pi}\psi^\star P_L\psi c - 2i\sqrt{\pi}\bar{\psi}P_R\bar{\psi}^\star c + B\bar{c}^\star]_{gf}}\,, \end{aligned} \qquad (12.123)$$

where

$$M_1 = -\frac{e^2}{2}\int\int\left[[(a-1)\partial_\mu + \tilde{\partial}_\mu]A^\mu\left(\frac{-1}{(a-1)\Box}\right)[(a-1)\partial_\nu + \tilde{\partial}_\nu]A^\nu\right]\,. \qquad (12.124)$$

This partition function only involves the gauge potentials, fermion fields, ghosts and corresponding antifields, but not the embedding θ field. M_1 is closely connected with $\alpha_1[A,\theta]$, the logarithm of the Jacobian associated with the gauge non-invariance of the fermionic measure. Indeed, one checks that under the gauge transformation $\delta A^\mu = \frac{1}{e}\partial^\mu\alpha$, M_1 transforms as $\delta M_1 = -\alpha_1[A,\alpha]$. Note that M_1 is non-local in the fields.

Let us decompose the argument of the exponential in (12.123) in the form

$$W[A,\psi,\bar{\psi},c,\bar{c};A^\star,\psi^\star,\bar{\psi}^\star,c^\star,\bar{c}^\star] = S[A,\psi,\bar{\psi},c,\bar{c};A^\star,\psi^\star,\bar{\psi}^\star,c^\star,\bar{c}^\star] + M_1[A]\ , \tag{12.125}$$

where

$$S = S_{ferm} + \int d^2x[A^\star_\mu \frac{1}{e}\partial^\mu c + 2i\sqrt{\pi}\psi^\star P_L\psi c - 2i\sqrt{\pi}\bar{\psi}P_R\bar{\psi}^\star c + B\bar{c}^\star]$$

satisfies the classical master equation $(S,S) = 0$. A simple computation shows that

$$(M_1, S) = e\int d^2x\, c(x)\ [(a-1)\partial_\mu + \tilde{\partial}_\mu]A^\mu(x)\ . \tag{12.126}$$

Furthermore, since M_1 does not depend on the antifields, we have that

$$i\Delta_{op}S = i\Delta_{op}W = \int d^2x\ c(x)\mathcal{A}(x)\ ,$$

where $\mathcal{A}$ is the anomaly (12.122). Hence

$$(M_1, S) = i\Delta_{op}S.$$

Recalling that an iterative solution to the quantum master equation is given by (12.101), where only the first two equations are relevant in the present case, since M_1 does not depend on the antifields, we conclude that the field-antifield action (12.125) is a solution to the quantum master equation in the original configuration space, including the ghosts. In this space it is however non-local. The absence of a local, but existence of a non-local solution to the quantum master equation has been taken in the literature to be a signature for a true anomaly, which cannot be eliminated by local counter terms [Troost 1990].

Note that in the computation of (M_1, S) from (12.124), gauge invariant terms did not contribute. The solution to (12.126) is thus not unique. As was observed by Batalin and Vilkovisky, we can always add gauge invariant terms to M_1 without affecting this equation.

The operator Δ_{op} in (12.121) is a very singular operator. Its action on W can be computed as a short distance limit using a Fujikawa [Fujikawa 1979] or Pauli Villars regularization [Troost 1990]. With the knowledge of (12.121) one could have proceeded directly by solving the quantum master equation. We would have thereby arrived at the non-local solution (12.125), with M_1 given by (12.124).

Our above analysis has shown that in the presence of a gauge anomaly the quantum master equation for the CSM has no local solution in the original space of fields. A local solution of the quantum master equation could however be obtained in an extended space; this solution involves the classical (gauge invariant) action S_{cl} and a (non-gauge invariant) "Wess-Zumino action" compensating the effects of the non-nvariance of the integration measure under BRST transformations. This action was just the BFT embedded effective quantum action.

Bosonized Chiral Schwinger model

It is interesting to compare the above embedded formulation of the CSM with the embedded version of its bosonized formulation. In this formulation the classical gauge invariance is of course again broken on quantum level, which now manifests itself in the action only.

Let us begin with the bosonization step. [25] The factor $e^{iW^{(L)}[A]}$ appearing in the partition function (12.114), with $W^{(L)}$ given by (12.115), can be rewritten as a Gaussian integral over a scalar field ϕ (bosonization),

$$Z = \int DAD\phi e^{iS_{bos}[A,\phi]} , \tag{12.127}$$

where S_{bos} is now the local action [Jackiw 1985]

$$S_{bos}[A,\phi] = \int d^2x \Big[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}ae^2 A_\mu A^\mu + \frac{1}{2}(\partial_\mu\phi)^2 + eA^\mu(\partial_\mu - \tilde{\partial}_\mu)\phi\Big] . \tag{12.128}$$

The action (12.128) describes a second class constrained system [Girotti 1986], reflecting the breaking of gauge invariance. Again, this system does not admit a priori a BV representation of the partition function. In order to apply the field-antifield formalism we shall thus have to turn it first into a first class system by embedding it into an extended space, following either the BFT procedure of chapter 7 (see section 15.4.4 of [Abdalla 2001]), or the procedure described in chapter 14 of that reference). The result is the equivalent effective action

$$S^{emb}_{bos}[A,\phi,\theta] = S_{bos}[A,\phi] + \alpha_1[A,\theta] , \tag{12.129}$$

which replaces (12.116). A little bit of algebra shows that this action is invariant under the extended gauge transformations

$$\delta A^\mu(x) = \frac{1}{e}\partial_\mu\alpha(x) , \quad \delta\phi(x) = -\alpha(x) , \quad \delta\theta(x) = \alpha(x) . \tag{12.130}$$

[25] For a more detailed study of the bosonized CSM see e.g. [Abdalla 2001].

Thus after bosonization and BFT embedding, both - the action and functional integration measure $DAD\phi D\theta$ - are invariant under the above transformation, while in the embedded fermionic formulation both, the measure and action separately broke the gauge symmetry of the partition function. We can thus write for the corresponding bosonized (gauge fixed) BV partition function,

$$Z = \int DB \int DAD\phi D\theta \int D\bar{c}Dc\; e^{iW[A,\phi,\theta,c,\bar{c};\; A^\star,\phi^\star,\theta^\star,c^\star,\bar{c}^\star;B]_{gf}} \tag{12.131}$$

with

$$W = S^{ext}_{bos}[A,\phi,\theta] + \int d^2x\,[A^\star_\mu \frac{1}{e}\partial^\mu c - \phi^\star c - \theta^\star c + B\bar{c}^\star]\,,$$

where, in line with our discussion in section 2, we have included a term $B\bar{c}^\star$ required for gauge fixing purposes. The antifields are again expressed in terms of the fields through functional derivatives of the "fermion gauge fixing functional" Ψ_L. One readily checks that in this bosonized form, W satisfies the quantum master equation, but also the classical master equation, $(W,W) = 0$, which is consistent with $\Delta_{op}W = 0$. The fields A_μ, ϕ in the second class formulation correspond to the gauge invariants $A^\theta_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta, \phi^\theta = \phi - \theta$ in the embedded formulation [Girotti 1989].

We therefore see that the existence of a quantum anomaly is not necessarily signalized by a non vanishing $\Delta_{op}W$. [26] Of course, by choosing the gauge $\theta = 0$, corresponding to the gauge fixing functional $\Psi_L = \int d^2x\; \bar{c}\,\theta$, we return to the partition function (12.127), which violates gauge invariance through the "classical" action. On the other hand, in the fermionic formulation discussed before, the violation is of quantum nature, since it is linked to the non-invariance of the fermionic measure, which is classically invariant. It is here where the anomaly manifests itself directly in that $\Delta_{op}W \neq 0$.

It is interesting to see what happens if we integrate (12.131) over θ, assuming the gauge fixing fermion not to depend on θ. The result is the partition function

$$Z = \int DB \int DAD\phi \int D\bar{c}Dc\; e^{iW[A,\phi,c,\bar{c};A^\star,\phi^\star,c^\star,\bar{c}^\star;B]_{gf}}$$

where now

$$W = S + M_1[A] \tag{12.132}$$

with

$$S = S_{bos} + \int d^2x\; [A^\star_\mu \frac{1}{e}\partial^\mu c - \phi^\star c + B\bar{c}^\star]$$

and M_1 given by the non-local expression (12.124). A simple computation shows that again $(W,W) = 0$, in accordance with the fact that $\Delta_{op}W$ in

[26] That $\Delta_{op}W = 0$ in this case is of course expected from our analysis in section 12.5.1, since the measure is now gauge invariant.

(12.91) vanishes trivially, since W does not involve any terms bilinear in the fields and corresponding antifields.

Let us summarize our findings. In both, the embedded fermionic and bosonic gauge invariant formulations, the quantum master equation is satisfied. There exists however a feature which distinguishes the two cases. In the bosonized case, where the action as well as integration measure are gauge invariant, $\Delta_{op}W = 0$ and the *classical* master equation holds. On the other hand, in the embedded fermionic case, where both the measure and action are not gauge invariant, the quantum master equation is satisfied with $\Delta_{op}W \neq 0$, and with $\Delta_{op}W$ given in terms of an integral over the consistent anomaly. This, in turn, is directly related to the non-invariance of the fermionic measure under chiral transformations. Hence a vanishing or non-vanishing contribution $\Delta_{op}W$ to the quantum master equation does not tell us whether the theory has a gauge anomaly or not. Common to both cases is however, that the solution to the quantum master equation obtained after integrating out the auxiliary embedding field in the partition function, is *non-local.* Also common to both formulations is that the original action is modified by a Wess-Zumino term. Both features are believed to be characteristic for theories, whose classical gauge invariance is broken by quantum effects.

Appendix A

Local Symmetries and Singular Lagrangians

In this appendix we recapitulate some well-known facts about how local symmetries are reflected in the existence of Ward identities and conservation laws. We will say nothing about how these symmetries can actually be detected. The following discussion is quite general, and involves only well-known facts. The presentation follows closely a review given in [Costa 1988], based on the pioneering work of Bergmann [Bergmann 1949]. Our main purpose is to show that any theory with a local symmetry implies a singular Lagrangian in the sense stated in chapter 2.

A.1 Local symmetry transformations

Let $\mathcal{L}(\varphi, \partial\varphi, \xi)$ be a Lagrangian density depending on a set of fields $\varphi(\xi) := \{\varphi^A(\xi)\}$ and their first field derivatives, $\partial\varphi := \{\partial_\mu \varphi\}$. Here ξ stands for the set $\{\xi^\mu\}$ $(\mu = 1, \cdots, r)$. The Euler-Lagrange equations of motion

$$E_A := \frac{\partial \mathcal{L}}{\partial \varphi^A} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^A)} = 0 \tag{A.1}$$

follow from the usual variational principle

$$\delta S[\varphi] = \delta \int_V d^r\xi \ \mathcal{L}(\varphi(\xi), \partial\varphi(\xi), \xi) = 0 \ ,$$

where V is a r-dimensional volume, and δS denotes the variation of S due to an arbitrary virtual displacement in the fields,

$$\varphi^A(\xi) \to \tilde{\varphi}^A(\xi) = \varphi^A(\xi) + \delta\varphi^A(\xi) \ , \tag{A.2}$$

subject to the boundary condition

$$\delta\varphi^A|_{\partial V} = 0 \ .$$

Since the variation does not involve a change in the ξ^μ, it is also referred to as a *vertical variation.* This variation commutes with the derivative operation: $\delta\partial_\mu - \partial_\mu\delta = 0$. The quantity E_A in (A.1) is the Euler derivative of $\mathcal{L}$. If the change in φ can be absorbed by a corresponding transformation of the arguments ξ^μ of φ, i.e., if

$$\tilde{\varphi}^A(\tilde{\xi}) = \varphi^A(\xi) \ , \quad \tilde{\xi}^\mu = \tilde{\xi}^\mu(\xi) \ , \tag{A.3}$$

then the transformation $\varphi(\xi) \to \tilde{\varphi}(\xi)$ is equivalent to a mere transformation of the argument. A more general class of transformations of this type, which also mixes the fields, are for instance rotations which are realized in the form

$$\tilde{\varphi}^A(\xi) = U^A_B(R)\varphi^B(R^{-1}\xi) \ ,$$

or

$$\tilde{\varphi}^A(\tilde{\xi}) = U^A_B(R)\varphi^B(\xi) \ , \ \tilde{\xi} = R\xi \ ,$$

where $R^{-1} = R^T$ and U^A_B is a unitary representation of the rotation group.

Consider now more general transformations,

$$\{\varphi^A(\xi)\} \to \{\tilde{\varphi}^A(\tilde{\xi})\} \ , \ \tilde{\xi} = \tilde{\xi}(\xi) \ ,$$

which may, or may not, be of the above type. If this transformation is invertible, then there exists a Lagrangian $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}(\tilde{\varphi}(\tilde{\xi}), \tilde{\partial}\tilde{\varphi}(\tilde{\xi}), \tilde{\xi})$ such that

$$\int_V d^r\xi \ \mathcal{L}(\varphi(\xi), \partial\varphi(\xi), \xi) = \int_{\tilde{V}} d^r\tilde{\xi} \ \tilde{\mathcal{L}}(\tilde{\varphi}(\tilde{\xi}), \tilde{\partial}\tilde{\varphi}(\tilde{\xi}), \tilde{\xi}) \ . \tag{A.4}$$

If $\delta\varphi|_{\partial V} = 0$ implies $\delta\tilde{\varphi}|_{\tilde{\partial}\tilde{V}} = 0$, then the variation of the rhs leads to the equations of motion

$$\frac{\partial\tilde{\mathcal{L}}}{\partial\tilde{\varphi}^A} - \tilde{\partial}_\mu \frac{\partial\tilde{\mathcal{L}}}{\partial(\tilde{\partial}_\mu\tilde{\varphi}^A)} = 0 \ , \tag{A.5}$$

which are fully equivalent to (A.1). Such a transformation will in general change the form of the equations of motion. If the form remains unchanged one speaks of a *symmetry-transformation.* As seen from (A.5), a sufficient condition for this to be the case is that the functional form of $\tilde{\mathcal{L}}$ and $\mathcal{L}$ is the same. Then the equality (A.4) implies that

$$J\left(\frac{\partial\tilde{\xi}}{\partial\xi}\right)\mathcal{L}(\tilde{\varphi}, \tilde{\partial}\tilde{\varphi}, \tilde{\xi}) - \mathcal{L}(\varphi, \partial\varphi, \xi) = \partial_\mu\Lambda^\mu \ , \tag{A.6}$$

where on the rhs ∂_μ acts on ξ, including any implicit dependence of Λ on ξ through the fields, and where $J(\partial\tilde{\xi}/\partial\xi)$ is the Jacobian for the transformation of the measure $d^r\tilde{\xi} \to d^r\xi$. For simplicity we have suppressed the argument $\tilde{\xi}$ of the fields $\tilde{\varphi}$. In the following we assume that the symmetry transformation in question belongs to a *continuous* group of transformations connected to the identity. In this case it is sufficient to consider infinitesimal transformations.

Let us introduce in addition to the virtual vertical displacement δ defined in (A.2), the following more general variation $\bar{\delta}F$, which also includes a transformation of the coordinates $\xi \to \tilde{\xi}(\xi)$,

$$F(\xi) \to \tilde{F}(\tilde{\xi}) = F(\xi) + \bar{\delta}F(\xi) \ . \tag{A.7}$$

The two variations δ and $\bar{\delta}$ are related by

$$\bar{\delta}F(\xi) = \delta F(\xi) + (\partial_\nu F(\xi))\delta\xi^\nu .$$

In particular one has

$$\begin{aligned} \bar{\delta}\varphi^A(\xi) &= \delta\varphi^A(\xi) + (\partial_\nu\varphi^A(\xi))\delta\xi^\nu \ , \\ \bar{\delta}(\partial_\mu\varphi^A(\xi)) &= \partial_\mu\delta\varphi^A(\xi) + (\partial_\nu\partial_\mu\varphi^A(\xi))\delta\xi^\nu \ , \end{aligned} \tag{A.8}$$

where we have made use of

$$\delta\partial_\mu - \partial_\mu\delta = 0. \tag{A.9}$$

On the other hand, the variation $\bar{\delta}$ does not commute with the partial derivatives. Instead we have that

$$\bar{\delta}(\partial_\mu\varphi^A) = \partial_\mu(\bar{\delta}\varphi^A) - (\partial_\nu\varphi^A)\partial_\mu\delta\xi^\nu \ . \tag{A.10}$$

Since for an infinitesimal transformation connected to the identity

$$\frac{\partial\tilde{\xi}^\mu}{\partial\xi^\nu} \simeq \delta^\mu_\nu + \partial_\nu\delta\xi^\mu \,,$$

one has for the Jacobian

$$J\left(\frac{\partial\tilde{\xi}}{\partial\xi}\right) \simeq 1 + \partial_\mu\delta\xi^\mu \ .$$

Hence for an infinitesimal symmetry transformation it follows from (A.6) that

$$(1 + \partial_\mu\delta\xi^\mu)\mathcal{L}(\varphi + \bar{\delta}\varphi, \partial\varphi + \bar{\delta}(\partial\varphi), \xi + \delta\xi) - \mathcal{L}(\varphi, \partial\varphi, \xi) = \partial_\mu\Lambda^\mu \,, \tag{A.11}$$

where the explicit form of Λ depends on the given Lagrangian. Expanded to first order, (A.11) can be written in terms of δ-variations in the form

$$\frac{\partial \mathcal{L}}{\partial \varphi^A}\delta\varphi^A + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^A)}(\partial_\mu \delta\varphi^A) + \frac{\partial \mathcal{L}}{\partial \varphi^A}(\partial_\mu \varphi^A)\delta\xi^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^A)}\partial_\nu(\partial_\mu \varphi^A)\delta\xi^\nu$$
$$+ \mathcal{L}\partial_\mu \delta\xi^\mu + \frac{\partial \mathcal{L}}{\partial \xi^\mu}\delta\xi^\mu = \partial_\mu \Lambda^\mu$$

or

$$E_A \delta\varphi^A + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^A)}\delta\varphi^A \right) + (\partial_\mu \mathcal{L})\delta\xi^\mu + \mathcal{L}\partial_\mu \delta\xi^\mu = \partial_\mu \Lambda^\mu \ . \tag{A.12}$$

Here $\partial_\mu \mathcal{L}$ denotes the derivative with respect to ξ^μ, *including the implicit ξ-dependence of the fields φ*. Hence (A.12) takes the form

$$E_A \delta\varphi^A + \partial_\mu \delta\rho^\mu \equiv 0 \tag{A.13}$$

where

$$\delta\rho^\mu = \mathcal{L}\delta\xi^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^A)}\delta\varphi^A - \Lambda^\mu \ , \tag{A.14}$$

and $\delta\varphi$ is defined, according to (A.7) and (A.8) in terms of the general variation $\bar{\delta}$ by [1]

$$\delta\varphi^A(\xi) = \tilde{\varphi}^A(\tilde{\xi}(\xi)) - \varphi^A(\xi) - \left(\partial_\mu \varphi^A(\xi)\right)\delta\xi^\mu$$

with $\delta\xi = \tilde{\xi} - \xi$. We emphasize that (A.13) is an identity which holds for arbitrary fields $\varphi^A(\xi)$, if the variations $\delta\xi^\mu$ and $\delta\varphi^A(\xi)$ are such that the action is invariant. This is true for both, global and local symmetries.

A.2 Bianchi identities and singular Lagrangians

A well-known theory with a non-trivial local symmetry is the $SU(N)$ Yang-Mills theory, where an infinitesimal gauge transformation of the potentials A_μ^a (a the colour) has the form

$$A_\mu^a \to A_\mu^a + f_{abc}A_\mu^c \epsilon^b - \frac{1}{g}\partial_\mu \epsilon^a$$

[1]Note that it was important to write (A.11) in terms of δ-variations in order to make use of (A.9).

Thus the variaton of the fields involves terms linear in the ξ-dependent parameters ϵ^a and their derivatives. We are thus led to consider local transformations of the form

$$\begin{aligned} \delta\xi^\mu &= \chi_l^\mu \epsilon^l(\xi)\,, \quad l = 1, \cdots, N\ , \\ \delta\varphi^A(\xi) &= \Phi_l^A \epsilon^l(\xi) + \Psi_l^{A\nu}\partial_\nu \epsilon^l(\xi)\,, \quad A = 1, \cdots, N_\varphi\ , \end{aligned} \tag{A.15}$$

which include the case of conformal primary fields. [2]

$$\begin{aligned} \delta\xi^\mu &= \chi_l^\mu \epsilon^l(\xi)\,, \quad l = 1, \cdots, N\ , \\ \delta\varphi^A(\xi) &= \Phi_l^A \epsilon^l(\xi) + \Psi_l^{A\nu}\partial_\nu \epsilon^l(\xi)\,, \quad A = 1, \cdots, N_\varphi\ . \end{aligned} \tag{A.16}$$

Correspondingly, Λ^μ and $\delta\rho^\mu$ will be of the form

$$\begin{aligned} \Lambda^\mu &= \lambda_l^\mu \epsilon^l(\xi) + \beta_l^{\mu\nu}\partial_\nu \epsilon^l(\xi)\ , \\ \delta\rho^\mu &= j_l^\mu \epsilon^l(\xi) + \gamma_l^{\mu\nu}\partial_\nu \epsilon^l(\xi)\ . \end{aligned} \tag{A.17}$$

Here Φ_l^A, $\Psi_l^{A\mu}$, λ_l^μ, j_l^μ, $\beta_l^{\mu\nu}$, $\gamma_l^{\mu\nu}$ will in general be functions of φ^A and ξ. Furthermore, χ_l^μ may depend on ξ^μ. Substituting (A.15) into (A.14) and comparing powers in $\partial\epsilon$ one is led to a relation among the above coefficient functions,

$$\gamma_l^{\mu\nu} = \Psi_l^{B\nu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^B)} - \beta_l^{\mu\nu}\,. \tag{A.18}$$

On the other hand, substituting (A.17) into (A.13) and comparing coefficients of $\partial\partial\epsilon$, one is led to the identities

$$\Psi_l^{A\mu}E_A + j_l^\mu + \partial_\nu\gamma^{\nu\mu} \equiv 0 \tag{A.19}$$

with

$$\gamma^{\mu\nu} = -\gamma^{\nu\mu}\,. \tag{A.20}$$

Note that on the space of solutions ($E_A = 0$) it follows from (A.19) that

$$j_l^\mu \equiv \partial_\nu\gamma_l^{\mu\nu} \quad (on - shell)\ ,$$

or using (A.20),

$$\partial_\mu j_l^\mu = 0 \quad (on - shell)\ .$$

Taking again account of the antisymmetry of $\gamma^{\mu\nu}$ in μ and ν, the identity (A.19) leads to

$$\partial_\mu j_\ell^\mu = -\partial_\mu(\Psi_\ell^{A\mu}E_A)\ , \quad (off - shell). \tag{A.21}$$

[2]For non-abelian $SU(N)$ gauge transformations we have, with $A \to (a,\mu)$, $l \to b$ and $\varphi^A \to A_a^\mu$, that $\Phi_l^A \to \Phi_b^{a\mu} = f_{abc}A_c^\mu$ and $\Psi_l^{A\nu} \to \Psi_b^{a\mu\nu} = -\frac{1}{g}\delta_b^a g^{\mu\nu}$.

Consider now a global transformation, i.e. with ϵ^l independent of ξ. Then (A.13) becomes

$$(\Phi_l^A E_A + \partial_\mu j_l^\mu)\epsilon^l \equiv 0 \ .$$

Hence we conclude that

$$\Phi_l^A E_A = -\partial_\mu j_l^\mu \tag{A.22}$$

so that (A.21) takes the form

$$\Phi_l^A E_A - \partial_\mu(\Psi_l^{A\mu} E_A) \equiv 0 \ , \quad (off\ shell) \ . \tag{A.23}$$

Equivalently, using (A.22), we have the conservation law

$$\partial_\mu \Theta_l^\mu = 0 \ ,$$

where

$$\Theta_l^\mu = \Psi_l^{A\mu} E_A + j_l^\mu \ .$$

The identities (A.23) are known as the *generalized Bianchi identities.*

In the following we shall assume that $\mathcal{L}$ is second order in the field derivatives. Then the Bianchi identities (A.23) imply a singular Lagrangian in the sense that the $\mu = \nu = 0$ component of the tensor

$$W_{AB}^{\mu\nu} = \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \varphi^A)\partial(\partial_\nu \varphi^B)} \ , \tag{A.24}$$

viewed as a matrix W (Hessian) with elements labeled by the pair (A, B), has eigenvectors with vanishing eigenvalue. In particular the determinant of W_{AB}^{00} vanishes. Indeed, it is not hard to see that, if the Lagrangian is of the form [3]

$$\mathcal{L} = \frac{1}{2} W_{AB}^{\mu\nu}(\varphi)\partial_\mu \varphi^A \partial_\nu \varphi^B - V(\varphi) \ , \tag{A.25}$$

then either $\vec{v}^{(l)} = \{\Psi_l^{A0}\}$ or $\vec{w}^{(l)} = \{\Phi_l^A\}$, appearing in the transformation law (A.15) for the fields φ^A, are such eigenvectors. Thus consider the Euler derivative (A.1). With (A.25) it has the form

$$E_A = \frac{\partial \mathcal{L}}{\partial \varphi^A} - \frac{\partial W_{AB}^{\mu\nu}}{\partial \varphi^C} \partial_\mu \varphi^C \partial_\nu \varphi^B - W_{AB}^{\mu\nu}(\varphi)\partial_\mu \partial_\nu \varphi^B \ . \tag{A.26}$$

Insert this expression in the Bianchi identity (A.23) and consider the terms involving different products of field derivatives of different orders. Since (A.23) is an identity, the corresponding coefficients, which are only functions of the

[3]Note that the contribution to $\mathcal{L}$ involving W_{AB}^{00} is just the kinetic term.

fields, must vanish. Consider the contribution proportional to $\partial_\mu \partial_\nu \partial_\lambda \varphi^A$. This leads to the identities

$$\Psi_l^{A[\mu} W_{AB}^{\nu\lambda]} \equiv 0 \; , \tag{A.27}$$

where the bracket denotes symmetrization. We thus have in particular that

$$\Psi_l^{A0} W_{AB}^{00} = 0 \; . \tag{A.28}$$

From here it follows that unless the vectors $\vec{v}^{(l)} \equiv \{\Psi_l^{A0}\}$ vanish for all l, there exists at least one eigenvector of W^{00} with vanishing eigenvalue, implying a singular matrix W_{AB}^{00}. In this case the corresponding Lagrangian is said to be "singular", or "degenerate". This matrix is just the Hessian introduced in chapter 2, i.e. (2.2). On the other hand, suppose that (A.28) possesses only the trivial solution $\Psi_l^{A0} = 0$. Since this equation holds in an arbitrary Lorentz frame, this implies that all $\Psi_l^{A\mu}$'s must vanish. Consequently, for a non-trivial symmetry transformation (A.15), there must exist at least one l for which not all the components of Φ_l^A (labeled by A) vanish. But by considering the contribution to (A.23) proportional to the second derivatives in the fields one is led to the identity $\Phi_\ell^A W_{AB}^{\mu\nu} = 0$, and in particular to

$$\Phi_l^A W_{AB}^{00} = 0 \; . \tag{A.29}$$

Hence, if $\Psi_l^{A\mu} = 0$ for all A, μ and l, then for a non-trivial symmetry transformation of the fields, there exists at least one value of l for which not all components of Φ_l^A (labeled by A) vanish. From (A.27) and (A.28) we therefore conclude that there exists at least one eigenvector of the Hessian matrix W_{AB}^{00} with vanishing eigenvalue. Thus if the action is invariant under a local symmetry transformation of the fields, then this implies a singular Lagrangian [Bergmann 1949]. This is the main statement we wanted to arrive at.

Appendix B

The BRST Charge of Rank One

In the following we prove that the BRST charge for theories of rank one is given by (11.98). For simplicity of notation we set $Q_{\mathcal{B}} = Q$. We now make the ansatz

$$Q = Q_0 + \sum_n \mathcal{Q}^{B_1 \cdots B_n}(q, p, \eta) \bar{\mathcal{P}}_{B_1} \cdots \bar{\mathcal{P}}_{B_n} \,, \tag{B.1}$$

with the "initial" condition

$$Q_0 = \eta^A G_A \,, \tag{B.2}$$

where $\{G_A = 0\}$ is the set of first class constraints. The coefficient functions $\mathcal{Q}^{B_1 \cdots B_n}$ can depend on the coordinates q_i, momenta p_i and on the Grassmann valued variables η^A. $\bar{\mathcal{P}}_A$ and η^A satisfy the graded Poisson algebra

$$\{\eta^A, \bar{\mathcal{P}}_A\} = -\delta^A_B.$$

Since Q carries Grassmann signature and ghost number 1, the coefficient functions $\mathcal{Q}^{B_1 \cdots B_n}$ will be polynomials of order $\mathcal{O}(n+1)$ in the η-variables. Hence with

$$\epsilon(\mathcal{Q}^{B_1 \cdots B_n}) = n + 1 \ (mod\, 2) \,, \quad gh(\mathcal{Q}^{B_1 \cdots B_n}) = n + 1$$

they will be of the form

$$\mathcal{Q}^{B_1 \cdots B_n} = Q^{B_1 \cdots B_n}_{A_1 \cdots A_{n+1}}(q, p) \eta^{A_1} \cdots \eta^{A_{n+1}} \,.$$

The inital condition (B.2) ensures that the BRST symmetry transformations implemented on the *non*-ghost variables are properly realized with the infinitesimal parameters replaced by η^A.

Assume Q to be of rank one:

$$Q = Q_0 + \mathcal{Q}^A \bar{\mathcal{P}}_A \ . \tag{B.3}$$

Taking account of the fact that Q_0 ($\mathcal{Q}^A$) has Grassmann parity $\epsilon = 1$ ($\epsilon = 0$) we have for the graded Poisson bracket $\{Q, Q\}$,

$$\begin{aligned}\{Q, Q\} = &\left(\{Q_0, Q_0\} + 2\mathcal{Q}^A\{Q_0, \bar{\mathcal{P}}_A\}\right) \\ &+ 2\left(\{Q_0, \mathcal{Q}^B\} + \mathcal{Q}^A\{\bar{\mathcal{P}}_A, \mathcal{Q}^B\}\right)\bar{\mathcal{P}}_B + \bar{\mathcal{P}}_A\{\mathcal{Q}^A, \mathcal{Q}^B\}\bar{\mathcal{P}}_B \ .\end{aligned}$$

The above expression has been ordered in ascending powers of $\bar{\mathcal{P}}$. Each term must vanish separately if Q is nilpotent. Making use of the expression for Q_0, and of the algebra of the constraints (11.97), one is led to the following conditions for $\mathcal{Q}^A$:

$$\begin{aligned} &i) \quad \left(\eta^B U^A_{BC}\eta^C - 2\mathcal{Q}^A\right) G_A = 0 \ , \\ &ii) \quad \left(\{Q_0, \mathcal{Q}^A\} - \mathcal{Q}^B \frac{\partial^{(l)} \mathcal{Q}^A}{\partial \eta^B}\right) \bar{\mathcal{P}}_A = 0 \ , \\ &iii) \quad \sum_{A,B} \bar{\mathcal{P}}_A\{\mathcal{Q}^A, \mathcal{Q}^B\}\bar{\mathcal{P}}_B = 0 \ . \end{aligned} \tag{B.4}$$

From *i)* we immediately deduce that

$$\mathcal{Q}^A = \frac{1}{2}\eta^B U^A_{BC}\eta^C \ . \tag{B.5}$$

Inserting this expression in *ii)* we are led to the requirement

$$\eta^A\eta^B\eta^C\left(\{G_A, U^D_{BC}\} - U^E_{AB}U^D_{EC}\right) = 0 \ . \tag{B.6}$$

That this equation is, in fact, identically satisfied, follows from the following Jacobi identity:

$$J \equiv \{G_A, \{G_B, G_C\}\} + cycl.\ perm = 0 \ . \tag{B.7}$$

Thus making use of the algebra of constraints we have that

$$\{G_A, U^D_{BC}G_D\} + cycl.\ perm.\ of\ ABC \equiv 0 \ ,$$

or

$$\left(\{G_A, U^D_{BC}\} - U^E_{BC}U^D_{EA}\right) G_D + cycl.\ perm.\ of\ ABC \equiv 0 \ . \tag{B.8}$$

Contracting this expression with $\eta^A\eta^B\eta^C$ and making use of the linear independence of the constraint functions G_A we retrieve (B.6), which is thus satisfied identically.

From (B.8) we obtain two further useful expressions which will be needed to show that *iii)* holds. Thus contracting this expression with η^B from the left and η^C from the right one readily verifies that

$$\{G_A, \hat{U}^B_C \eta^C\} \equiv -2\{G_C\eta^C, \hat{U}^B_A\} - U^B_{AD}\hat{U}^D_C\eta^C - 2\hat{U}^B_D\hat{U}^D_A \ , \tag{B.9}$$

where

$$\hat{U}^A_B = \eta^C U^A_{CB} \ . \tag{B.10}$$

By further contracting the identity (B.9) with η^A from the left we obtain

$$\{G\eta, \hat{U}^B_C\eta^C\} = -\hat{U}^B_D\hat{U}^D_C\eta^C \ , \tag{B.11}$$

where

$$G\eta \equiv G_A\eta^A \ .$$

Finally we show that *iii)* in (B.4) also holds identically.

We first note that because of the antisymmetry of the Poisson bracket in $\{\mathcal{Q}^A, \mathcal{Q}^B\}$, and the definition (B.5), the proof of *iii)* is equivalent to showing that

$$\{\hat{U}^A_B\eta^B, \hat{U}^C_D\eta^D\} = 0 \ . \tag{B.12}$$

Indeed, consider the following Jacobi identity

$$\tilde{J} \equiv \{U^A_{EB}, \{G_C, G_D\}\} + \{G_C, \{G_D, U^A_{EB}\}\} + \{G_D, \{U^A_{EB}, G_C\}\} \equiv 0 \ .$$

Contract this expression from the left with $\eta^E\eta^B\eta^C\eta^D$. This leads to

$$\tilde{J} = \{\hat{U}^A_B\eta^B, \{G\eta, G\eta\}\} + 2\{G\eta, \{G\eta, \hat{U}^A_B\eta^B\}\} \equiv 0 \ . \tag{B.13}$$

Note that $\{\ ,\ \}$ always stands for the *graded* Poisson bracket. Making use of the algebra of constraints and of (B.11) this expression becomes

$$\{\hat{U}^A_C\eta^C, \hat{U}^B_D\eta^D\}G_B + \{\hat{U}^A_C\eta^C, G_B\}\hat{U}^B_D\eta^D - 2\{G\eta, \hat{U}^A_B\hat{U}^B_C\eta^C\} \equiv 0 \ .$$

Making further use of (B.9) one then finds that

$$\{\hat{U}^A_B\eta^B, \hat{U}^C_D\eta^D\}G_C + 2U^A_{CE}\hat{U}^E_B\eta^B\hat{U}^C_D\eta^D + 2\hat{U}^A_E\hat{U}^E_C\hat{U}^C_D\eta^D - 2\{G\eta, \hat{U}^B_C\eta^C\}\hat{U}^A_B \equiv 0.$$

Because of (B.11) the last contribution reduces to $-2\hat{U}^A_B\hat{U}^B_D\hat{U}^D_C\eta^C$ which cancels the third term. What concerns the second term, this contribution vanishes because of the antisymmetry of U^A_{CE} in C and E. Hence we conclude that

$$\{\hat{U}^A_B\eta^B, \hat{U}^C_D\eta^D\}G_C \equiv 0 \ ,$$

which implies (B.12) because of the linear independence of the constraints. Concluding, we have verified, that with $\mathcal{Q}^A$ given by (B.5), all the conditions (B.4) are satisfied. Hence (B.3) is a nilpotent operator.

Appendix C

BRST Hamiltonian of Rank One

In the following we construct the BRST Hamiltonian for a theory of rank one and show that it is given by (11.101). As in the case of the BRST charge, we make an ansatz in terms of polynomials in $\bar{\mathcal{P}}_A$. Since $H_{\mathcal{B}}$ carries vanishing Grassmann signature and ghost number, this ansatz is of the form

$$H_{\mathcal{B}} = H + \sum_n \mathcal{H}^{B_1\cdots B_n}(q,p,\eta)\bar{\mathcal{P}}_{B_1}\cdots\bar{\mathcal{P}}_{B_n} \ ,$$

where H either stands for the canonical Hamiltonian evaluated on the primary surface Γ_P, or on the full constrained surface Γ, and where $\mathcal{H}^{B_1\ldots B_n}$ are tensors in the Grassmann valued η^A-variables,

$$\mathcal{H}^{B_1\cdots B_n} = H^{B_1\cdots B_n}_{A_1\cdots A_n}(q,p)\eta^{A_1}\cdots\eta^{A_n} \ ,$$

carrying ghost number and Grassmann parity n. Assume $H_{\mathcal{B}}$ to be of rank one:

$$H_{\mathcal{B}} = H + \mathcal{H}^A\bar{\mathcal{P}}_A \ ,$$

where

$$\mathcal{H}^A = H^A_B\eta^B \ .$$

Then the requirement that $H_{\mathcal{B}}$ be BRST invariant, i.e. $\{Q, H_{\mathcal{B}}\} = 0$, with Q given by (B.3), leads to the following equation

$$\begin{aligned}
&\left(\{H,G_A\}\eta^A - \mathcal{H}^A G_A\right)\\
&+ \left(\{H,\mathcal{Q}^A\} - \{\mathcal{H}^A,G_B\}\eta^B + \mathcal{H}^B\{\bar{\mathcal{P}}_B,\mathcal{Q}^A\} - \{\mathcal{H}^A,\bar{\mathcal{P}}_B\}\mathcal{Q}^B\right)\bar{\mathcal{P}}_A\\
&- \bar{\mathcal{P}}_B\{\mathcal{H}^B,\mathcal{Q}^A\}\bar{\mathcal{P}}_A = 0 \ .
\end{aligned}$$

The expression is ordered in ascending powers of $\bar{\mathcal{P}}$. Hence we are led to the following conditions:

$$
\begin{aligned}
&i) \quad \{H, G_A\}\eta^A - \mathcal{H}^A G_A = 0 \ , \\
&ii) \quad \left(\{H, \mathcal{Q}^A\} - \{\mathcal{H}^A, G_B\}\eta^B + \mathcal{H}^B\{\bar{\mathcal{P}}_B, \mathcal{Q}^A\} - \{\mathcal{H}^A, \bar{\mathcal{P}}_B\}\mathcal{Q}^B\right)\bar{\mathcal{P}}_A = 0 \ , \\
&iii) \quad \bar{\mathcal{P}}_A\{\mathcal{H}^A, \mathcal{Q}^B\}\bar{\mathcal{P}}_B = 0 \ .
\end{aligned}
\tag{C.1}
$$

Making use of the algebra (11.100), condition *i)* immediately leads to

$$
\mathcal{H}^A = \eta^B V_B^A \ . \tag{C.2}
$$

Consider now condition *ii)*. Written out explicitly, using (B.5) and (C.2), one finds that

$$
\begin{aligned}
[\frac{1}{2}\eta^B\{H, U_{BC}^A\}\eta^C - \eta^C\{V_C^A, G_B\}\eta^B - \eta^C V_C^B U_{BD}^A \eta^D \\
+ \frac{1}{2}V_B^A U_{CD}^B \eta^C \eta^D]\bar{\mathcal{P}}_A = 0 \, .
\end{aligned}
\tag{C.3}
$$

Consider further the following Jacobi identity

$$
\{G_A, \{G_B, H\}\} + cycl.\ perm. \equiv 0 \ .
$$

Making use of the algebra of the constraints (11.97) one finds that

$$
2\eta^B\{V_B^C, G_A\}\eta^A + 2\eta^B U_{BD}^C V_A^D \eta^A + \eta^B\{H, U_{AB}^C\}\eta^A + \eta^B U_{AB}^D V_D^C \eta^A \equiv 0 \, .
$$

The left hand side of this identity is proportional to the quantity appearing within brackets in (C.3). Hence we conclude that condition *ii)* is satisfied identically. This is not the case for condition *iii)* in (C.1). By inserting the expressions (C.2) and (B.5) in (C.1), *iii)* is seen to be equivalent to the requirement stated in theorem 2 in section 11.5.

Appendix D

The FV Principal Theorem

In this appendix we prove that the partition function

$$Z_\Psi = \int D\xi D\omega \; e^{i\int_{t_1}^{t_2} dt \; [\dot{\xi}^\ell \omega_\ell - H_{\mathcal{B}}(\xi,\omega) - \{\Psi, Q_{\mathcal{B}}\}]} \; , \tag{D.1}$$

where ξ and ω stand for the collection of coordinates and conjugate momenta, including ghosts, does not depend on $\Psi(\xi, \omega)$, if the following conditions hold:

i) $Q_{\mathcal{B}}$ is a nilpotent operator carrying ghost number $+1$, i.e.,

$$\{Q_{\mathcal{B}}, Q_{\mathcal{B}}\} = 0 \; ; \; gh(Q_{\mathcal{B}}) = 1 \; . \tag{D.2}$$

ii) $Q_{\mathcal{B}}$ fulfills the boundary condition

$$\left[\left(\frac{\partial Q_{\mathcal{B}}}{\partial \omega_\ell} \right) \omega_\ell - Q_{\mathcal{B}} \right]_{t_1}^{t_2} = 0 \; . \tag{D.3}$$

iii) The Poisson bracket of $H_{\mathcal{B}}$ with $Q_{\mathcal{B}}$ vanishes,

$$\{Q_{\mathcal{B}}, H_{\mathcal{B}}\} = 0 \; . \tag{D.4}$$

Proof

Consider the partition function (D.1) and the change of variables

$$\begin{aligned} \xi^\ell \to \tilde{\xi}^\ell &= \xi^\ell + \bar{\epsilon}\{Q_{\mathcal{B}}, \xi^\ell\} \; , \\ \omega_\ell \to \tilde{\omega}_\ell &= \omega_\ell + \bar{\epsilon}\{Q_{\mathcal{B}}, \omega_\ell\} \; , \end{aligned} \tag{D.5}$$

where $\bar{\epsilon}$ is an infinitesimal Grassmann valued functional of $\{\xi^\ell\}$ and $\{\omega_\ell\}$:

$$\bar{\epsilon}[\xi, \omega] = -i \int_{t_1}^{t_2} d\tau \; \Delta\Psi(\xi(\tau), \omega(\tau)) \; . \tag{D.6}$$

Here $\Delta\Psi(\xi, \omega)$ is an arbitrary infinitesimal Grassmann valued function of the coordinates and momenta with ghost number -1. Under this change of variables Z_Ψ remains unchanged. Furthermore, the argument of the exponential in (D.1) remains invariant. Indeed, consider first the change in $H_\mathcal{B}$ induced by (D.5). One readily verifies that

$$H_\mathcal{B}(\tilde{\xi}, \tilde{\omega}) = H_\mathcal{B}(\xi, \omega) + \bar{\epsilon}\{Q_\mathcal{B}, H_\mathcal{B}\} + \mathcal{O}(\bar{\epsilon}^2) \; .$$

The second term on the rhs vanishes because of (D.4), by assumption, so that

$$H_\mathcal{B}(\tilde{\xi}, \tilde{\omega}) = H_\mathcal{B}(\xi, \omega) \; .$$

Consider next the BRST exact term in (D.1),

$$\{\Psi, Q_\mathcal{B}\}(\tilde{\xi}, \tilde{\omega}) = \{\Psi, Q_\mathcal{B}\}(\xi, \omega) + \bar{\epsilon}\{Q_\mathcal{B}, \{\Psi, Q_\mathcal{B}\}\} + \mathcal{O}(\bar{\epsilon}^2) \; .$$

Again, the second term on the rhs vanishes because of (D.2) and the Jacobi identity. Furthermore, if condition ii) holds, then the kinetic term contribution to the action is also invariant under the transformations (D.5), as we have already shown in chapter 11, since $\bar{\epsilon}$ does not depend on time. Hence we conclude that the action in (D.1) is invariant under the transformation (D.5). This is not the case for the integration measure, since $\bar{\epsilon}$ in (D.6) depends on the coordinates and momenta, so that the transformation (D.5) is not canonical. This is the crucial observation.

Let us compute the Jacobian of the (infinitesimal) transformation. To simplify our expressions let us collect the coordinates ξ^ℓ and momenta ω_ℓ at a fixed time into a vector $\vec{\eta}(t)$:

$$\vec{\eta} = (\xi^1, \cdots, \xi^N, \omega_1, \cdots, \omega_N) \; .$$

Let us think for the moment of time to be discretized. For an infinitesimal transformation of variables,

$$\delta\eta_{\underline{\ell}} = \bar{\epsilon}\{Q, \eta_{\underline{\ell}}\} \; ,$$

we then have

$$\prod_{\underline{\ell}} d\eta_{\underline{\ell}} = J^{-1} \prod_{\underline{\ell}} d\tilde{\eta}_{\underline{\ell}} \; , \tag{D.7}$$

where "time" has now been absorbed into the label "$\underline{\ell}$". For an infinitesimal transformation with mixed bosonic and fermionic variables the "superjacobian" J is given by [1]

$$J = 1 + \sum_{\underline{\ell}} (-1)^{\epsilon_{\eta_\ell}} \frac{\delta^{(l)}(\delta\eta_{\underline{\ell}})}{\delta\eta_{\underline{\ell}}},$$

where $\delta\eta_{\underline{\ell}} = \tilde{\eta}_{\underline{\ell}} - \eta_{\underline{\ell}}$. or

$$J = 1 + \sum_{\underline{\ell}} (-1)^{\epsilon_{\xi_\ell}} \left[\frac{\delta^{(l)}}{\delta\xi^{\underline{\ell}}} (\bar{\epsilon}\{Q, \xi^{\underline{\ell}}\}) + \frac{\delta^{(l)}}{\delta\omega_{\underline{\ell}}} (\bar{\epsilon}\{Q, \omega_{\underline{\ell}}\}) \right] ,$$

since $\epsilon_{\xi_\ell} = \epsilon_{\omega_\ell}$. Writing out the graded Poisson brackets, defined in (11.43), and performing the differentiations, J takes the form,

$$J = 1 + \sum_{\underline{\ell}} \bar{\epsilon} \left[-\frac{\delta^{(l)}}{\delta\xi^{\underline{\ell}}} \left(\frac{\delta^{(r)}Q}{\delta\omega_{\underline{\ell}}} \right) + (-1)^{\epsilon_{\xi_\ell}} \frac{\delta^{(l)}}{\delta\omega_{\underline{\ell}}} \left(\frac{\delta^{(r)}Q}{\delta\xi^{\ell}} \right) \right]$$

$$+ \sum_{\underline{\ell}} (-1)^{\epsilon_{\xi_\ell}} \left[-\frac{\delta^{(l)}\bar{\epsilon}}{\delta\xi^{\underline{\ell}}} \frac{\delta^{(r)}Q}{\delta\omega_{\underline{\ell}}} + (-1)^{\epsilon_{\xi_\ell}} \frac{\delta^{(l)}\bar{\epsilon}}{\delta\omega_{\underline{\ell}}} \frac{\delta^{(r)}Q}{\delta\xi^{\underline{\ell}}} \right]$$

Making use of

$$\frac{\delta^{(l)}}{\delta\xi^{\ell}} \left(\frac{\delta^{(r)}Q_{\mathcal{B}}}{\delta\omega_\ell} \right) = (-1)^{\epsilon_{\xi_\ell}} \frac{\delta^{(l)}}{\delta\omega_\ell} \left(\frac{\delta^{(r)}Q_{\mathcal{B}}}{\delta\xi^{\ell}} \right) ,$$

we see that the first sum in the above expression for J vanishes. Since $Q_{\mathcal{B}}$ and $\bar{\epsilon}$ have both Grassmann signature -1, one then finds that

$$J \approx 1 - \sum_{\underline{\ell}} (-1)^{\epsilon_{\xi_\ell}} \left[\frac{\delta^{(r)}Q}{\delta\xi^{\underline{\ell}}} \frac{\delta^{(l)}\bar{\epsilon}}{\delta\omega_{\underline{\ell}}} - (-1)^{\epsilon_{\xi_\ell}} \frac{\delta^{(r)}Q}{\delta\omega_{\underline{\ell}}} \frac{\delta^{(l)}\bar{\epsilon}}{\delta\xi^{\underline{\ell}}} \right] \approx 1 - \{Q_{\mathcal{B}}, \bar{\epsilon}\} .$$

Returning to continuous time, the Jacobian J for the transformation of the functional measure in (D.1) is given by

$$J \approx 1 + \int_{t_1}^{t_2} dt \ \{\Delta\Psi, Q_{\mathcal{B}}\} \approx e^{i \int_{t_1}^{t_2} dt \ \{\Delta\Psi, Q_{\mathcal{B}}\}} ,$$

where we made use of the fact that $\{\Delta\Psi, Q_{\mathcal{B}}\} = \{Q_{\mathcal{B}}, \Delta\Psi\}$, since $\Delta\Psi$ and $Q_{\mathcal{B}}$ are both Grassmann valued. Hence, upon making the change of variables (D.7)

[1] The phase in the sum takes into account that for fermions the Jacobian is actually the inverse of the usual bosonic one. See also in this connection the comments in subsection 5.1 of chapter 12.

in the measure, and making use of the invariance of $H_{\mathcal{B}}(\xi, \omega)$ and the BRST exact term, the partition function (D.1) takes the following form :

$$Z_\Psi = \int D\tilde{\xi} D\tilde{\omega} \; e^{-i\int dt \{\Delta\Psi, Q_{\mathcal{B}}\}(\tilde{\xi},\tilde{\omega})} \; e^{i\int_{t_1}^{t_2} dt \;\left(\dot{\tilde{\xi}}^\ell \tilde{\omega}_\ell - H_{\mathcal{B}}(\tilde{\xi},\tilde{\omega}) - \{\Psi, Q_{\mathcal{B}}\}\right)} \; .$$

Combining the two exponentials we are thus left with the statement

$$Z_\Psi = Z_{\Psi + \Delta\Psi} \; .$$

This completes the proof.

Appendix E

BRST Quantization of $SU(3)$ Yang-Mills Theory in α-gauges

In this appendix we perform a BRST analysis of the $SU(N)$ Yang-Mills theory, analogous to that described in section 4 of chapter 11, by implementing the so-called α-gauges. In contrast to the case discussed in that section, the α-gauges cannot be realized as a single gauge condition on the fields. Nevertheless, the phase space representation of the partition function can again be cast into the form of (11.103) with H_U given by (11.104) together with (11.106).

Consider the following class of covariant gauge conditions:

$$\partial^\mu A_\mu^a(x) - \lambda^a(x) = 0 \ .$$

Following again the Faddeev-Popov procedure described in section 4 of chapter 11, we are led to the following form for the partition function,

$$Z = \int \mathcal{D}A \prod_{x,a} \delta(\partial^\mu A_\mu^a(x) - \lambda^a(x)) e^{i\int d^4x [\mathcal{L} + \bar{c}_a \partial^\mu \mathcal{D}_\mu^{ab} c^b]} \ ,$$

where $\mathcal{L}$ is given by (11.59). The trick, due to 't Hooft, consists in averaging this expression over $\lambda^a(x)$ with a Gaussian weight $e^{i\int d^4x \ \frac{\vec{\lambda}^2}{2\alpha}}$, making use of the fact that gauge invariant correlation functions (i.e., observables) do not depend on $\lambda^a(x)$. One is then led to

$$Z = \int \mathcal{D}A \ e^{i\int d^4x \ \mathcal{L}_{gf}} \tag{E.1}$$

with the gauge fixed Lagrangian density

$$\mathcal{L}_{gf} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + \frac{1}{2\alpha}\sum_a(\partial^\mu A^a_\mu)^2 - (\partial_\mu \bar{c}_a)\mathcal{D}^\mu_{ab}c^b\,, \tag{E.2}$$

where we have dropped an irrelevant four-divergence. Here $\mathcal{D}^\mu_{ab}$ is the covariant derivative defined in (11.65) and $F^a_{\mu\nu}$ the field strength tensor (11.60). The partition function (E.1) describes an unconstrained system.

The Euler–Lagrange equations, corresponding to stationary points of the action, read

$$\begin{aligned} \mathcal{D}^{ab}_\mu F^{\mu\nu}_b + \frac{1}{\alpha}\partial^\nu(\partial\cdot A^a) - g f_{abc}(\partial^\nu \overline{c}_b)c^c = 0 \quad , \\ \partial_\mu(\mathcal{D}^\mu_{ab}c^b) = 0\,, \quad \mathcal{D}^\mu_{ab}(\partial_\mu \overline{c}_b) = 0\ , \end{aligned} \tag{E.3}$$

which now replace (11.82). Note that there is no equivalent equation to $\partial^\mu A^a_\mu = 0$. Note also that the A^a_0 field has now been promoted to a dynamical (propagating) field. Although the Lagrangian (E.2) breaks the original gauge invariance, the associated action still possesses a BRST symmetry. The corresponding infinitesimal transformations of the fields read

$$\begin{aligned} \delta_B A^a_\mu &= \epsilon\mathcal{D}^{ab}_\mu c^b\ , \\ \delta_B \overline{c}_a &= -\epsilon\left(\frac{\partial\cdot A^a}{\alpha}\right)\ , \\ \delta_B c^a &= -\epsilon\frac{g}{2}f_{abc}c^b c^c\ . \end{aligned} \tag{E.4}$$

The verification of the invariance of the action under this transformation involves some tedious, but straightforward algebra, and makes use of the Jacobi identity.

i) The BRST Charge in configuration space

We next obtain the conserved BRST charge which will generate the BRST transformations in phase space. Following the same procedure as described in section 4 of chapter 11, one finds that the conserved BRST current reads,

$$\mathcal{J}^\mu_{\mathcal{B}} = F^{\mu\nu}_a\mathcal{D}^{ab}_\nu c^b + \frac{g}{2}f_{abc}(\partial^\mu\overline{c}_a)c^b c^c - \left(\frac{\partial\cdot A^a}{\alpha}\right)\mathcal{D}^\mu_{ab}c^b\ .$$

Hence the corresponding conserved BRST-charge in configuration space is given by

$$Q_{\mathcal{B}} = \int d^3x\mathcal{J}^0_{\mathcal{B}} = \int d^3x\ \left(F^a_{0i}\mathcal{D}^i_{ab}c^b - \frac{1}{\alpha}(\partial\cdot A^a)\mathcal{D}^0_{ab}c^b + \frac{g}{2}f_{abc}(\partial_0\overline{c}_a)c^b c^c\right)\quad . \tag{E.5}$$

So far our discussion was carried out in configuration space. We next construct the Hamiltonian and the phase space representation of the BRST charge.

ii) Hamiltonian

To obtain the phase space version of (E.1) we first construct the BRST Hamiltonian from L_{gf} by a Legendre transformation. The canonical momenta conjugate to the fields A^0_a, A^i_a, c^a and $\bar{c}_a$ are given respectively by

$$\begin{aligned} \pi_0^a &= \frac{1}{\alpha}(\partial^\mu A_\mu^a) \ , \quad \pi_i^a = F_{i0}^a \\ \bar{P}_a &= \partial_0 \bar{c}_a \ , \quad P^a = -D^0_{ab} c^b \ . \end{aligned} \tag{E.6}$$

One then finds for the Hamiltonian

$$H_U = H_0 + \int d^3x \left[\bar{P}_a P^a - g f_{abc} \bar{P}_a c^b A_0^c - (\partial^i A_i^a)\pi_0^a + (\partial_i \bar{c}_a)\mathcal{D}^i_{ab} c^b\right] \ , \tag{E.7}$$

where

$$H_0 = \int d^3x \left[\frac{\alpha}{2}\pi_0^a\pi_0^a + \frac{1}{2}\pi_i^a\pi_i^a + \frac{1}{4}F_{ij}^a F_a^{ij} - A_0^a \mathcal{D}^i_{ab}\pi_i^b\right]$$

is the canonical Hamiltonian.

iii) BRST charge in phase space

Consider the expression (E.5). Expressed in terms of the canonical variables (E.6), it takes the form

$$Q_{\mathcal{B}} = \int d^3x \ \left[c^a(\mathcal{D}^i_{ab}\pi_i^b) + \pi_0^a P^a - \frac{g}{2} f_{abc}\bar{P}_a c^b c^c\right] \quad .$$

One verifies that with (11.70) $Q_{\mathcal{B}}$ can be written in the form (11.98). With the generalized Poisson brackets defined by (11.43), one readily verifies that $\{Q_B, Q_B\} = 0$, (or on operator level, the nilpotency of the BRST charge: $\hat{Q}_B^2 = 0$). The BRST transformations of the fields written in Poisson bracket form read

$$\begin{aligned} sA_\mu^a &= \{Q_B, A_\mu^a\} = \mathcal{D}_\mu^{ab} c^b \quad , \\ sc^a &= \{Q_B, c^a\} = -\frac{g}{2} f_{abc} c^b c^c \ , \\ s\bar{c}_a &= \{Q_B, \overline{c}^a\} = -\pi_0^a = -\frac{1}{\alpha}(\partial^\mu A_\mu^a) \quad , \end{aligned} \tag{E.8}$$

where we have defined the operator s by (11.22). From (E.6) and (E.8) one then finds that the corresponding variations in the momenta are given by

$$s\pi_0^a = 0 \ ,$$

$$\begin{aligned}
s\pi_i^a &= g f_{abd}\pi_i^b c^d \ , \\
s\bar{P}_a &= -\mathcal{D}^i_{ab}\pi_i^b - g f_{abd}\bar{P}_b c^d \ , \\
\hat{s}P^a &= 0 \ .
\end{aligned}$$

These are symmetries of the Hamilton equations of motion (i.e., on shell symmetries).

iv) BRST form of the unitarizing Hamiltonian

One now verifies that the unitarizing Hamiltonian (E.7) can also be written in the form

$$H_U = H + \int d^3x \ \left(\eta^A V_A^B \bar{\mathcal{P}}_B + \{\Psi, Q_B\}\right) \ ,$$

where H is either the canonical Hamiltonian H_0 of the classical theory, evaluated on the primary constrained surface, or the canonical Hamiltonian H_0' evaluated on the full constrained surface. V_A^B are the structure functions (11.72) if $H = H$, or vanish if $H = H'$. Thus for example, with the following choice for the "fermion gauge fixing function",

$$\Psi = \bar{c}_a(\partial^i A_i^a - \frac{\alpha}{2}\pi_0^a) + \bar{P}_a A_0^a \ ,$$

one finds that

$$\{\Psi(x), Q_{\mathcal{B}}\} = -\bar{c}_a\partial_i\mathcal{D}^i_{ab}c^b - \pi_0^a(\partial^i A_i^a - \frac{\alpha}{2}\pi_0^a) - A_0^a\mathcal{D}^i_{ab}\pi_i^b + \bar{P}_a P^a + g f_{abc}A_0^a c^b \bar{P}_c \ .$$

Hence, upon taking into account the fact that the structure functions V_A^B vanish if $H = H_0'$, and that the canonical Hamiltonian density of the $SU(3)$ Yang-Mills theory evaluated on the full constraint surface is given by

$$\mathcal{H}' = \frac{1}{2}\pi_0^a\pi_0^a + \frac{1}{4}F_{ij}^a F_a^{ij} \ ,$$

we conclude that, although the so-called "α-gauges" cannot be realized as conditions on the fields, the partition function has the form appearing in the FV-princpal theorem (11.103).

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